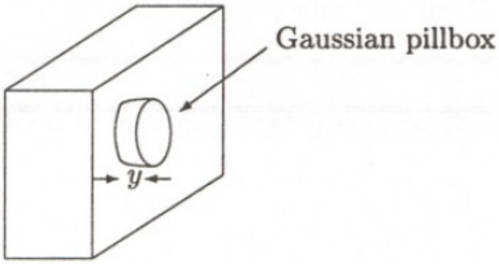
Problem 2.17

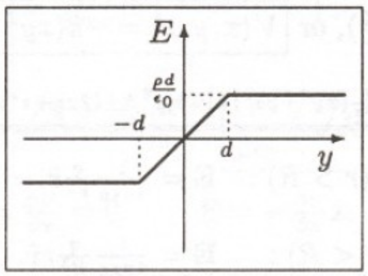
On the xz plane $E = 0$ by symmetry. Set up a Gaussian "pillbox" with one face in this plane and the other at y.



$$\int \mathbf{E}\cdot d\mathbf{a} = E\cdot A = \frac{1}{\epsilon_0}Q_{\text{enc}} = \frac{1}{\epsilon_0}Ay\rho;$$

$$\boxed{\mathbf{E} = \frac{\rho}{\epsilon_0}y\,\hat{\mathbf{y}}} \text{ (for } |y| < d).$$

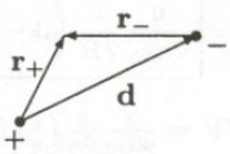
$Q_{\text{enc}} = \frac{1}{\epsilon_0}Ad\rho \Rightarrow \boxed{\mathbf{E} = \dfrac{\rho}{\epsilon_0}d\,\hat{\mathbf{y}}}$ (for $y > d$).



Problem 2.18

From Prob. 2.12, the field inside the positive sphere is $\mathbf{E}_+ = \frac{\rho}{3\epsilon_0}\mathbf{r}_+$, where $\mathbf{r}_+$ is the vector from the positive center to the point in question. Likewise, the field of the negative sphere is $-\frac{\rho}{3\epsilon_0}\mathbf{r}_-$. So the *total* field is

$$\mathbf{E} = \frac{\rho}{3\epsilon_0}(\mathbf{r}_+ - \mathbf{r}_-)$$



But (see diagram) $\mathbf{r}_+ - \mathbf{r}_- = \mathbf{d}$. So $\boxed{\mathbf{E} = \dfrac{\rho}{3\epsilon_0}\mathbf{d}.}$

Problem 2.21

$$V(r) = -\int_\infty^r \mathbf{E}\cdot d\mathbf{l}. \quad \begin{cases} \text{Outside the sphere } (r > R): & \mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{r}}. \\ \text{Inside the sphere } (r < R): & \mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{R^3}r\hat{\mathbf{r}}. \end{cases}$$

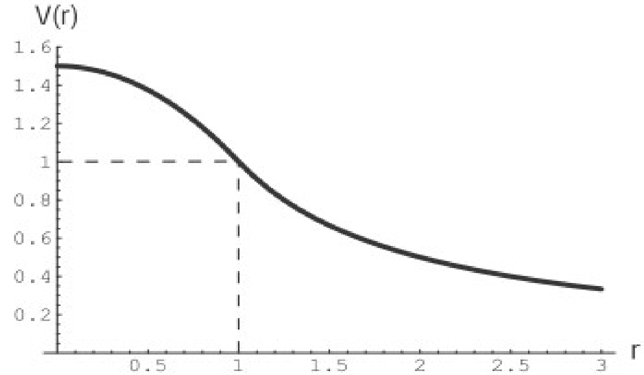
So for $r > R$: $V(r) = -\int_\infty^r \left(\frac{1}{4\pi\epsilon_0}\frac{q}{\bar{r}^2}\right)d\bar{r} = \frac{1}{4\pi\epsilon_0}q\left(\frac{1}{\bar{r}}\right)\Big|_\infty^r = \boxed{\dfrac{q}{4\pi\epsilon_0}\dfrac{1}{r},}$

and for $r < R$: $V(r) = -\int_\infty^R \left(\frac{1}{4\pi\epsilon_0}\frac{q}{\bar{r}^2}\right)d\bar{r} - \int_R^r \left(\frac{1}{4\pi\epsilon_0}\frac{q}{R^3}\bar{r}\right)d\bar{r} = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{R} - \frac{1}{R^3}\left(\frac{r^2-R^2}{2}\right)\right]$

$$= \boxed{\frac{q}{4\pi\epsilon_0}\frac{1}{2R}\left(3 - \frac{r^2}{R^2}\right).}$$

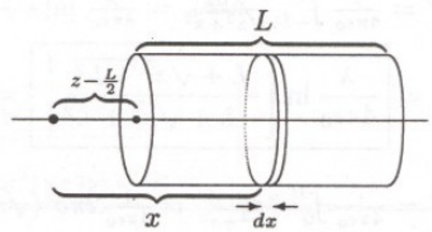
When $r > R$, $\nabla V = \frac{q}{4\pi\epsilon_0}\frac{\partial}{\partial r}\left(\frac{1}{r}\right)\hat{\mathbf{r}} = -\frac{q}{4\pi\epsilon_0}\frac{1}{r^2}\hat{\mathbf{r}}$, so $\mathbf{E} = -\nabla V = \frac{q}{4\pi\epsilon_0}\frac{1}{r^2}\hat{\mathbf{r}}$. ✓

When $r < R$, $\nabla V = \frac{q}{4\pi\epsilon_0}\frac{1}{2R}\frac{\partial}{\partial r}\left(3 - \frac{r^2}{R^2}\right)\hat{\mathbf{r}} = \frac{q}{4\pi\epsilon_0}\frac{1}{2R}\left(-\frac{2r}{R^2}\right)\hat{\mathbf{r}} = -\frac{q}{4\pi\epsilon_0}\frac{r}{R^3}\hat{\mathbf{r}}$; so $\mathbf{E} = -\nabla V = \frac{1}{4\pi\epsilon_0}\frac{q}{R^3}r\hat{\mathbf{r}}$. ✓



In the figure, r is in units of R, and $V(r)$ is in units of $\frac{q}{4\pi\epsilon_0 R}$.

Problem 2.27



Cut the cylinder into slabs, as shown in the figure, and use result of Prob. 2.25c, with $z \to x$ and $\sigma \to \rho\, dx$:

$$V = \frac{\rho}{2\epsilon_0} \int_{z-L/2}^{z+L/2} \left(\sqrt{R^2+x^2} - x\right) dx$$

$$= \frac{\rho}{2\epsilon_0} \frac{1}{2} \left[x\sqrt{R^2+x^2} + R^2 \ln(x + \sqrt{R^2+x^2}) - x^2 \right]\Big|_{z-L/2}^{z+L/2}$$

$$= \boxed{\frac{\rho}{4\epsilon_0} \left\{ \left(z+\frac{L}{2}\right)\sqrt{R^2+\left(z+\frac{L}{2}\right)^2} - \left(z-\frac{L}{2}\right)\sqrt{R^2+\left(z-\frac{L}{2}\right)^2} + R^2 \ln\left[\frac{z+\frac{L}{2}+\sqrt{R^2+\left(z+\frac{L}{2}\right)^2}}{z-\frac{L}{2}+\sqrt{R^2+\left(z-\frac{L}{2}\right)^2}}\right] - 2zL \right\}}.$$

(*Note:* $-\left(z+\frac{L}{2}\right)^2 + \left(z-\frac{L}{2}\right)^2 = -z^2 - zL - \frac{L^2}{4} + z^2 - zL + \frac{L^2}{4} = -2zL$.)

$$\mathbf{E} = -\boldsymbol{\nabla} V = -\hat{\mathbf{z}}\frac{\partial V}{\partial z} = -\frac{\hat{\mathbf{z}}\rho}{4\epsilon_0}\left\{ \sqrt{R^2+\left(z+\frac{L}{2}\right)^2} + \frac{\left(z+\frac{L}{2}\right)^2}{\sqrt{R^2+\left(z+\frac{L}{2}\right)^2}} - \sqrt{R^2+\left(z-\frac{L}{2}\right)^2} - \frac{\left(z-\frac{L}{2}\right)^2}{\sqrt{R^2+\left(z-\frac{L}{2}\right)^2}} \right.$$

$$\left. + R^2 \underbrace{\left[\frac{1+\frac{z+\frac{L}{2}}{\sqrt{R^2+\left(z+\frac{L}{2}\right)^2}}}{z+\frac{L}{2}+\sqrt{R^2+\left(z+\frac{L}{2}\right)^2}} - \frac{1+\frac{z-\frac{L}{2}}{\sqrt{R^2+\left(z-\frac{L}{2}\right)^2}}}{z-\frac{L}{2}+\sqrt{R^2+\left(z-\frac{L}{2}\right)^2}} \right]}_{\frac{1}{\sqrt{R^2+\left(z+\frac{L}{2}\right)^2}} - \frac{1}{\sqrt{R^2+\left(z-\frac{L}{2}\right)^2}}} - 2L \right\}$$

$$\mathbf{E} = -\frac{\hat{\mathbf{z}}\rho}{4\epsilon_0}\left\{ 2\sqrt{R^2+\left(z+\frac{L}{2}\right)^2} - 2\sqrt{R^2+\left(z-\frac{L}{2}\right)^2} - 2L \right\}$$

$$= \boxed{\frac{\rho}{2\epsilon_0}\left[L - \sqrt{R^2+\left(z+\frac{L}{2}\right)^2} + \sqrt{R^2+\left(z-\frac{L}{2}\right)^2} \right]\hat{\mathbf{z}}.}$$

Problem 2.32

(a) $W = \frac{1}{2}\int \rho V d\tau$. From Prob. 2.21 (or Prob. 2.28): $V = \frac{\rho}{2\epsilon_0}\left(R^2 - \frac{r^2}{3}\right) = \frac{1}{4\pi\epsilon_0}\frac{q}{2R}\left(3 - \frac{r^2}{R^2}\right)$

$$W = \frac{1}{2}\rho\frac{1}{4\pi\epsilon_0}\frac{q}{2R}\int_0^R \left(3 - \frac{r^2}{R^2}\right) 4\pi r^2 dr = \frac{q\rho}{4\epsilon_0 R}\left[3\frac{r^3}{3} - \frac{1}{R^2}\frac{r^5}{5}\right]\Bigg|_0^R = \frac{q\rho}{4\epsilon_0 R}\left(R^3 - \frac{R^3}{5}\right)$$

$$= \frac{q\rho}{5\epsilon_0}R^2 = \frac{qR^2}{5\epsilon_0}\frac{q}{\frac{4}{3}\pi R^3} = \boxed{\frac{1}{4\pi\epsilon_0}\left(\frac{3}{5}\frac{q^2}{R}\right).}$$

(b) $W = \frac{\epsilon_0}{2}\int E^2 d\tau$. *Outside* $(r > R)$ $\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{r}}$; *Inside* $(r < R)$ $\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{R^3}r\hat{\mathbf{r}}$.

$$\therefore W = \frac{\epsilon_0}{2}\frac{1}{(4\pi\epsilon_0)^2}q^2\left\{\int_R^\infty \frac{1}{r^4}(r^2 4\pi\, dr) + \int_0^R \left(\frac{r}{R^3}\right)^2 (4\pi r^2 dr)\right\}$$

$$= \frac{1}{4\pi\epsilon_0}\frac{q^2}{2}\left\{\left(-\frac{1}{r}\right)\Bigg|_R^\infty + \frac{1}{R^6}\left(\frac{r^5}{5}\right)\Bigg|_0^R\right\} = \frac{1}{4\pi\epsilon_0}\frac{q^2}{2}\left(\frac{1}{R} + \frac{1}{5R}\right) = \frac{1}{4\pi\epsilon_0}\frac{3}{5}\frac{q^2}{R}.\checkmark$$

(c) $W = \frac{\epsilon_0}{2}\left\{\oint_S V\mathbf{E}\cdot d\mathbf{a} + \int_{\mathcal{V}} E^2 d\tau\right\}$, where $\mathcal{V}$ is large enough to enclose all the charge, but otherwise arbitrary. Let's use a sphere of radius $a > R$. Here $V = \frac{1}{4\pi\epsilon_0}\frac{q}{r}$.

$$W = \frac{\epsilon_0}{2}\left\{\int_{r=a}\left(\frac{1}{4\pi\epsilon_0}\frac{q}{r}\right)\left(\frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\right) r^2\sin\theta\, d\theta\, d\phi + \int_0^R E^2 d\tau + \int_R^a \left(\frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\right)^2 (4\pi r^2 dr)\right\}$$

$$= \frac{\epsilon_0}{2}\left\{\frac{q^2}{(4\pi\epsilon_0)^2}\frac{1}{a}4\pi + \frac{q^2}{(4\pi\epsilon_0)^2}\frac{4\pi}{5R} + \frac{1}{(4\pi\epsilon_0)^2}4\pi q^2\left(-\frac{1}{r}\right)\Bigg|_R^a\right\}$$

$$= \frac{1}{4\pi\epsilon_0}\frac{q^2}{2}\left\{\frac{1}{a} + \frac{1}{5R} - \frac{1}{a} + \frac{1}{R}\right\} = \frac{1}{4\pi\epsilon_0}\frac{3}{5}\frac{q^2}{R}.\checkmark$$

As $a \to \infty$, the contribution from the surface integral $\left(\frac{1}{4\pi\epsilon_0}\frac{q^2}{2a}\right)$ goes to zero, while the volume integral $\left(\frac{1}{4\pi\epsilon_0}\frac{q^2}{2a}\left(\frac{6a}{5R} - 1\right)\right)$ picks up the slack.

Problem 2.49

(a) $\boxed{\mathbf{E} = \frac{1}{4\pi\epsilon_0}\int \frac{\rho\hat{\mathscr{r}}}{\mathscr{r}^2}\left(1 + \frac{\mathscr{r}}{\lambda}\right)e^{-\mathscr{r}/\lambda}d\tau.}$

(b) $\boxed{\text{Yes.}}$ The field of a point charge at the origin is radial and symmetric, so $\nabla\times\mathbf{E} = 0$, and hence this is also true (by superposition) for any *collection* of charges.

(c)
$$V = -\int_\infty^r \mathbf{E}\cdot d\mathbf{l} = -\frac{1}{4\pi\epsilon_0}q\int_\infty^r \frac{1}{r^2}\left(1 + \frac{r}{\lambda}\right)e^{-r/\lambda}dr$$

$$= \frac{1}{4\pi\epsilon_0}q\int_r^\infty \frac{1}{r^2}\left(1 + \frac{r}{\lambda}\right)e^{-r/\lambda}dr = \frac{q}{4\pi\epsilon_0}\left\{\int_r^\infty \frac{1}{r^2}e^{-r/\lambda}dr + \frac{1}{\lambda}\int_r^\infty \frac{1}{r}e^{-r/\lambda}dr\right\}.$$

Now $\int \frac{1}{r^2} e^{-r/\lambda} dr = -\frac{e^{-r/\lambda}}{r} - \frac{1}{\lambda}\int \frac{e^{-r/\lambda}}{r} dr \longleftarrow$ exactly right to kill the last term. Therefore

$$V(r) = \frac{q}{4\pi\epsilon_0}\left\{ -\frac{e^{-r/\lambda}}{r}\bigg|_r^\infty \right\} = \boxed{\frac{q}{4\pi\epsilon_0}\frac{e^{-r/\lambda}}{r}.}$$

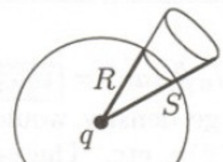
(d)

$$\oint_S \mathbf{E}\cdot d\mathbf{a} = \frac{1}{4\cancel{\pi}\epsilon_0} q \frac{1}{\cancel{R^2}}\left(1+\frac{R}{\lambda}\right) e^{-R/\lambda}\, 4\cancel{\pi}\,\cancel{R^2} = \frac{q}{\epsilon_0}\left(1+\frac{R}{\lambda}\right)e^{-R/\lambda}.$$

$$\int_{\mathcal{V}} V\,d\tau = \frac{q}{4\cancel{\pi}\epsilon_0}\int_0^R \frac{e^{-r/\lambda}}{r} r^2\, 4\cancel{\pi}\, dr = \frac{q}{\epsilon_0}\int_0^R r e^{-r/\lambda} dr = \frac{q}{\epsilon_0}\left[\frac{e^{-r/\lambda}}{(1/\lambda)^2}\left(-\frac{r}{\lambda}-1\right)\right]_0^R$$

$$= \lambda^2 \frac{q}{\epsilon_0}\left\{ -e^{-R/\lambda}\left(1+\frac{R}{\lambda}\right)+1\right\}.$$

$$\therefore \oint_S \mathbf{E}\cdot d\mathbf{a} + \frac{1}{\lambda^2}\int_{\mathcal{V}} V\,d\tau = \frac{q}{\epsilon_0}\left\{\left(1+\frac{R}{\lambda}\right)e^{-R/\lambda} - \left(1+\frac{R}{\lambda}\right)e^{-R/\lambda}+1\right\} = \frac{q}{\epsilon_0}. \quad \text{qed}$$



(e) Does the result in (d) hold for a *non*spherical surface? Suppose we make a "dent" in the sphere—pushing a patch (area $R^2\sin\theta\,d\theta\,d\phi$) from radius R out to radius S (area $S^2\sin\theta\,d\theta\,d\phi$).

$$\Delta\oint \mathbf{E}\cdot d\mathbf{a} = \frac{q}{4\pi\epsilon_0}\left\{\frac{1}{S^2}\left(1+\frac{S}{\lambda}\right)e^{-S/\lambda}(S^2\sin\theta\,d\theta\,d\phi) - \frac{1}{R^2}\left(1+\frac{R}{\lambda}\right)e^{-R/\lambda}(R^2\sin\theta\,d\theta\,d\phi)\right\}$$

$$= \frac{q}{4\pi\epsilon_0}\left[\left(1+\frac{S}{\lambda}\right)e^{-S/\lambda} - \left(1+\frac{R}{\lambda}\right)e^{-R/\lambda}\right]\sin\theta\,d\theta\,d\phi.$$

$$\Delta\frac{1}{\lambda^2}\int V\,d\tau = \frac{1}{\lambda^2}\frac{q}{4\pi\epsilon_0}\int \frac{e^{-r/\lambda}}{r} r^2 \sin\theta\,,dr\,d\theta\,d\phi = \frac{1}{\lambda^2}\frac{q}{4\pi\epsilon_0}\sin\theta\,d\theta\,d\phi\int_R^S r e^{-r/\lambda} dr$$

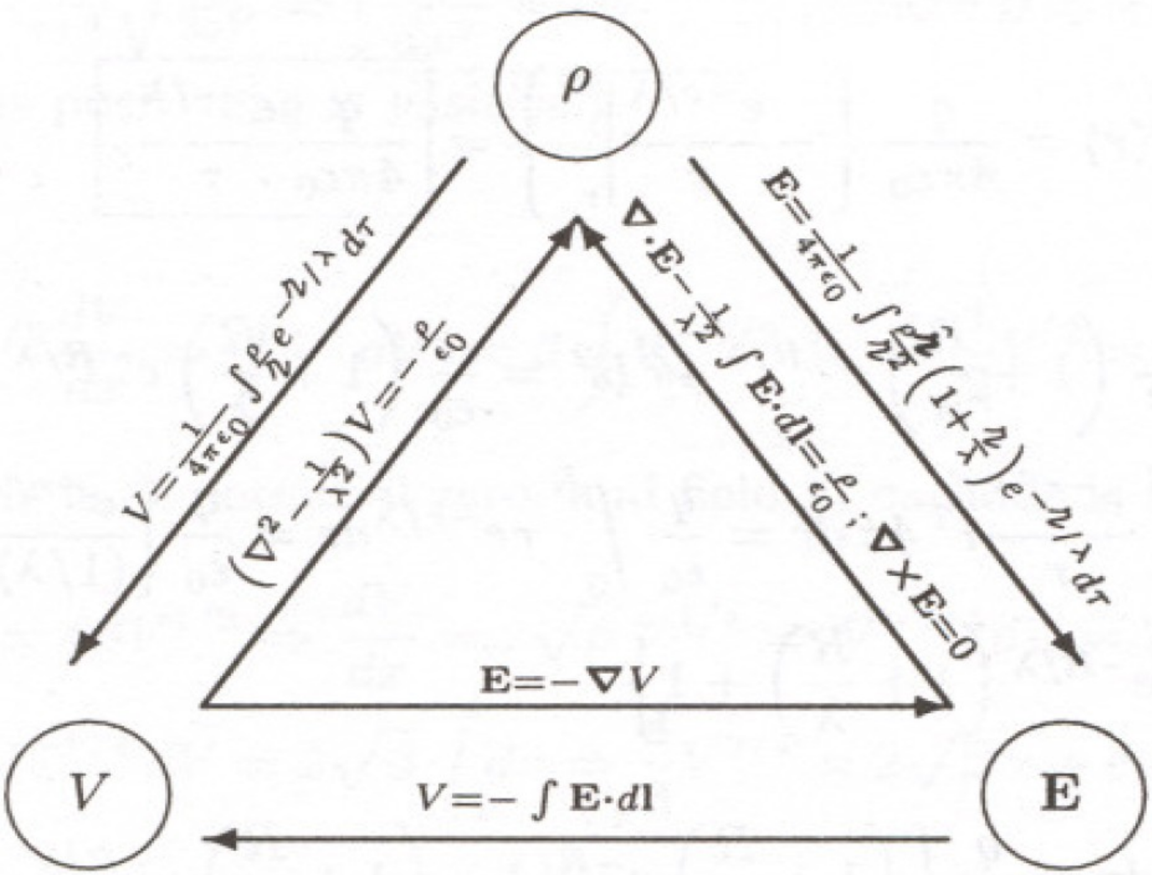
$$= -\frac{q}{4\pi\epsilon_0}\sin\theta\,d\theta\,d\phi\left(e^{-r/\lambda}\left(1+\frac{r}{\lambda}\right)\right)\bigg|_R^S$$

$$= -\frac{q}{4\pi\epsilon_0}\left[\left(1+\frac{S}{\lambda}\right)e^{-S/\lambda} - \left(1+\frac{R}{\lambda}\right)e^{-R/\lambda}\right]\sin\theta\,d\theta\,d\phi.$$

So the change in $\frac{1}{\lambda^2}\int V\,d\tau$ exactly compensates for the change in $\oint \mathbf{E}\cdot d\mathbf{a}$, and we get $\frac{1}{\epsilon_0}q$ for the total using the dented sphere, just as we did with the perfect sphere. Any closed surface can be built up by successive distortions of the sphere, so the result holds for all shapes. By superposition, if there are many charges inside, the total is $\frac{1}{\epsilon_0}Q_{\text{enc}}$. Charges *outside* do not contribute (in the argument above we found that  for this volume $\oint \mathbf{E}\cdot d\mathbf{a} + \frac{1}{\lambda^2}\int V\,d\tau = 0$—and, again, the sum is not changed by distortions of the surface, as long as q remains outside). So the new "Gauss's Law" holds for *any* charge configuration.

(f) In differential form, "Gauss's law" reads: $\boxed{\nabla\cdot\mathbf{E} + \frac{1}{\lambda^2}V = \frac{1}{\epsilon_0}\rho,}$ or, putting it all in terms of $\mathbf{E}$:

$\nabla\cdot\mathbf{E} - \frac{1}{\lambda^2}\int \mathbf{E}\cdot d\mathbf{l} = \frac{1}{\epsilon_0}\rho$. Since $\mathbf{E} = -\nabla V$, this also yields "Poisson's equation": $-\nabla^2 V + \frac{1}{\lambda^2}V = \frac{1}{\epsilon_0}\rho$.


ρ
V
E
V=−(1/(4πε0)) ∫ (ρ/r) e^(−r/λ) dτ
(∇² − 1/λ²)V = −ρ/ε0
∇·E − (1/λ²) ∫ E·dl = ρ/ε0 ; ∇×E=0
E = (1/(4πε0)) ∫ (ρ r̂/r²)(1 + r/λ) e^(−r/λ) dτ
E=−∇V
V=−∫E·dl