Problem 1.31

$T(\mathbf{b}) = 1+4+2 = 7;\ T(\mathbf{a}) = 0.\ \Rightarrow \boxed{T(\mathbf{b}) - T(\mathbf{a}) = 7.}$

$\nabla T = (2x+4y)\hat{\mathbf{x}} + (4x+2z^3)\hat{\mathbf{y}} + (6yz^2)\hat{\mathbf{z}};\ \nabla T \cdot d\mathbf{l} = (2x+4y)dx + (4x+2z^3)dy + (6yz^2)dz$

(a) Segment 1: $x: 0 \to 1,\ y = z = dy = dz = 0.\ \int \nabla T \cdot d\mathbf{l} = \int_0^1 (2x)\, dx = x^2\big|_0^1 = 1.$
Segment 2: $y: 0 \to 1,\ x = 1,\ z = 0,\ dx = dz = 0.\ \int \nabla T \cdot d\mathbf{l} = \int_0^1 (4)\, dy = 4y\big|_0^1 = 4.$
Segment 3: $z: 0 \to 1,\ x = y = 1,\ dx = dy = 0.\ \int \nabla T \cdot d\mathbf{l} = \int_0^1 (6z^2)\, dz = 2z^3\big|_0^1 = 2.$
$\Big\}\ \int_{\mathbf{a}}^{\mathbf{b}} \nabla T \cdot d\mathbf{l} = 7.$ ✓

(b) Segment 1: $z: 0 \to 1,\ x = y = dx = dy = 0.\ \int \nabla T \cdot d\mathbf{l} = \int_0^1 (0)\, dz = 0.$
Segment 2: $y: 0 \to 1,\ x = 0,\ z = 1,\ dx = dz = 0.\ \int \nabla T \cdot d\mathbf{l} = \int_0^1 (2)\, dy = 2y\big|_0^1 = 2.$
Segment 3: $x: 0 \to 1,\ y = z = 1,\ dy = dz = 0.\ \int \nabla T \cdot d\mathbf{l} = \int_0^1 (2x+4)\, dx$
$= (x^2+4x)\big|_0^1 = 1+4 = 5.$
$\Big\}\ \int_{\mathbf{a}}^{\mathbf{b}} \nabla T \cdot d\mathbf{l} = 7.$ ✓

(c) $x: 0 \to 1,\ y = x,\ z = x^2,\ dy = dx, dz = 2x\, dx.$

$\nabla T \cdot d\mathbf{l} = (2x+4x)dx + (4x+2x^6)dx + (6xx^4)2x\, dx = (10x + 14x^6)dx.$

$\int_{\mathbf{a}}^{\mathbf{b}} \nabla T \cdot d\mathbf{l} = \int_0^1 (10x + 14x^6)dx = (5x^2 + 2x^7)\big|_0^1 = 5 + 2 = 7.$ ✓

Problem 1.32

$\nabla \cdot \mathbf{v} = y + 2z + 3x$

$$\begin{aligned}
\int (\nabla \cdot \mathbf{v}) d\tau &= \int (y + 2z + 3x)\, dx\, dy\, dz = \iint \left\{ \int_0^2 (y + 2z + 3x)\, dx \right\} dy\, dz \\
&\qquad \hookrightarrow \left[(y+2z)x + \tfrac{3}{2}x^2\right]_0^2 = 2(y+2z) + 6 \\
&= \int \left\{ \int_0^2 (2y + 4z + 6) dy \right\} dz \\
&\qquad \hookrightarrow \left[y^2 + (4z+6)y\right]_0^2 = 4 + 2(4z+6) = 8z + 16 \\
&= \int_0^2 (8z + 16) dz = (4z^2 + 16z)\big|_0^2 = 16 + 32 = \boxed{48.}
\end{aligned}$$

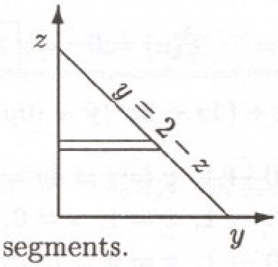
Numbering the surfaces as in Fig. 1.29:

(i) $d\mathbf{a} = dy\, dz\, \hat{\mathbf{x}}, x = 2.\ \mathbf{v} \cdot d\mathbf{a} = 2y\, dy\, dz.\ \int \mathbf{v} \cdot d\mathbf{a} = \iint 2y\, dy\, dz = 2y^2\big|_0^2 = 8.$
(ii) $d\mathbf{a} = -dy\, dz\, \hat{\mathbf{x}}, x = 0.\ \mathbf{v} \cdot d\mathbf{a} = 0.\ \int \mathbf{v} \cdot d\mathbf{a} = 0.$
(iii) $d\mathbf{a} = dx\, dz\, \hat{\mathbf{y}}, y = 2.\ \mathbf{v} \cdot d\mathbf{a} = 4z\, dx\, dz.\ \int \mathbf{v} \cdot d\mathbf{a} = \iint 4z\, dx\, dz = 16.$
(iv) $d\mathbf{a} = -dx\, dz\, \hat{\mathbf{y}}, y = 0.\ \mathbf{v} \cdot d\mathbf{a} = 0.\ \int \mathbf{v} \cdot d\mathbf{a} = 0.$
(v) $d\mathbf{a} = dx\, dy\, \hat{\mathbf{z}}, z = 2.\ \mathbf{v} \cdot d\mathbf{a} = 6x\, dx\, dy.\ \int \mathbf{v} \cdot d\mathbf{a} = 24.$
(vi) $d\mathbf{a} = -dx\, dy\, \hat{\mathbf{z}}, z = 0.\ \mathbf{v} \cdot d\mathbf{a} = 0.\ \int \mathbf{v} \cdot d\mathbf{a} = 0.$
$\Rightarrow \int \mathbf{v} \cdot d\mathbf{a} = 8 + 16 + 24 = 48$ ✓

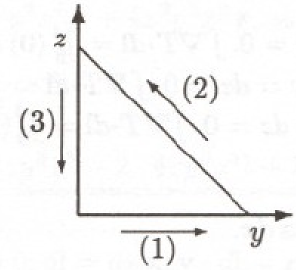
Problem 1.33

$\nabla \times \mathbf{v} = \hat{\mathbf{x}}(0 - 2y) + \hat{\mathbf{y}}(0 - 3z) + \hat{\mathbf{z}}(0 - x) = -2y\,\hat{\mathbf{x}} - 3z\,\hat{\mathbf{y}} - x\,\hat{\mathbf{z}}.$
$d\mathbf{a} = dy\, dz\, \hat{\mathbf{x}}$, if we agree that the path integral shall run counterclockwise. So
$(\nabla \times \mathbf{v}) \cdot d\mathbf{a} = -2y\, dy\, dz.$

$$\int(\nabla\times\mathbf{v})\cdot d\mathbf{a} = \int\left\{\int_0^{2-z}(-2y)dy\right\}dz$$
$$\hookrightarrow y^2\Big|_0^{2-z} = -(2-z)^2$$
$$= -\int_0^2(4-4z+z^2)dz = -\left(4z-2z^2+\frac{z^3}{3}\right)\Big|_0^2$$
$$= -\left(8-8+\frac{8}{3}\right) = \boxed{-\frac{8}{3}}$$



Meanwhile, $\mathbf{v}\cdot d\mathbf{l} = (xy)dx + (2yz)dy + (3zx)dz$. There are three segments.



(1) $x = z = 0;\ dx = dz = 0.\ y: 0 \to 2.\ \int\mathbf{v}\cdot d\mathbf{l} = 0.$
(2) $x = 0;\ z = 2-y;\ dx = 0,\ dz = -dy,\ y: 2 \to 0.\ \mathbf{v}\cdot d\mathbf{l} = 2yz\,dy.$
$\int\mathbf{v}\cdot d\mathbf{l} = \int_2^0 2y(2-y)dy = -\int_0^2(4y-2y^2)dy = -\left(2y^2-\frac{2}{3}y^3\right)\Big|_0^2 = -\left(8-\frac{2}{3}\cdot 8\right) = -\frac{8}{3}.$
(3) $x = y = 0;\ dx = dy = 0;\ z: 2 \to 0.\ \mathbf{v}\cdot d\mathbf{l} = 0.\ \int\mathbf{v}\cdot d\mathbf{l} = 0.$ So $\oint\mathbf{v}\cdot d\mathbf{l} = -\frac{8}{3}$. ✓

Problem 1.34

By Corollary 1, $\int(\nabla\times\mathbf{v})\cdot d\mathbf{a}$ should equal $\frac{4}{3}$. $\nabla\times\mathbf{v} = (4z^2-2x)\hat{\mathbf{x}} + 2z\,\hat{\mathbf{z}}$.

(i) $d\mathbf{a} = dy\,dz\,\hat{\mathbf{x}},\ x = 1;\ y, z: 0 \to 1.\ (\nabla\times\mathbf{v})\cdot d\mathbf{a} = (4z^2-2)dy\,dz;\ \int(\nabla\times\mathbf{v})\cdot d\mathbf{a} = \int_0^1(4z^2-2)dz$
$= \left(\frac{4}{3}z^3-2z\right)\Big|_0^1 = \frac{4}{3}-2 = -\frac{2}{3}.$
(ii) $d\mathbf{a} = -dx\,dy\,\hat{\mathbf{z}},\ z = 0;\ x, y: 0 \to 1.\ (\nabla\times\mathbf{v})\cdot d\mathbf{a} = 0;\ \int(\nabla\times\mathbf{v})\cdot d\mathbf{a} = 0.$
(iii) $d\mathbf{a} = dx\,dz\,\hat{\mathbf{y}},\ y = 1;\ x, z: 0 \to 1.\ (\nabla\times\mathbf{v})\cdot d\mathbf{a} = 0;\ \int(\nabla\times\mathbf{v})\cdot d\mathbf{a} = 0.$
(iv) $d\mathbf{a} = -dx\,dz\,\hat{\mathbf{y}},\ y = 0;\ x, z: 0 \to 1.\ (\nabla\times\mathbf{v})\cdot d\mathbf{a} = 0;\ \int(\nabla\times\mathbf{v})\cdot d\mathbf{a} = 0.$
(v) $d\mathbf{a} = dx\,dy\,\hat{\mathbf{z}},\ z = 1;\ x, y: 0 \to 1.\ (\nabla\times\mathbf{v})\cdot d\mathbf{a} = 2\,dx\,dy;\ \int(\nabla\times\mathbf{v})\cdot d\mathbf{a} = 2.$
$\Rightarrow \int(\nabla\times\mathbf{v})\cdot d\mathbf{a} = -\frac{2}{3}+2 = \frac{4}{3}$. ✓