Problem 1.13

$\boldsymbol{\mathscr{r}} = (x-x')\,\hat{\mathbf{x}} + (y-y')\,\hat{\mathbf{y}} + (z-z')\,\hat{\mathbf{z}}; \quad \mathscr{r} = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}.$

(a) $\nabla(\mathscr{r}^2) = \frac{\partial}{\partial x}[(x-x')^2 + (y-y')^2 + (z-z')^2]\,\hat{\mathbf{x}} + \frac{\partial}{\partial y}()\,\hat{\mathbf{y}} + \frac{\partial}{\partial z}()\,\hat{\mathbf{z}} = 2(x-x')\,\hat{\mathbf{x}} + 2(y-y')\,\hat{\mathbf{y}} + 2(z-z')\,\hat{\mathbf{z}} = 2\boldsymbol{\mathscr{r}}.$

(b) $\nabla(\frac{1}{\mathscr{r}}) = \frac{\partial}{\partial x}[(x-x')^2 + (y-y')^2 + (z-z')^2]^{-\frac{1}{2}}\,\hat{\mathbf{x}} + \frac{\partial}{\partial y}()^{-\frac{1}{2}}\,\hat{\mathbf{y}} + \frac{\partial}{\partial z}()^{-\frac{1}{2}}\,\hat{\mathbf{z}}$

$= -\frac{1}{2}()^{-\frac{3}{2}}2(x-x')\,\hat{\mathbf{x}} - \frac{1}{2}()^{-\frac{3}{2}}2(y-y')\,\hat{\mathbf{y}} - \frac{1}{2}()^{-\frac{3}{2}}2(z-z')\,\hat{\mathbf{z}}$

$= -()^{-\frac{3}{2}}[(x-x')\,\hat{\mathbf{x}} + (y-y')\,\hat{\mathbf{y}} + (z-z')\,\hat{\mathbf{z}}] = -(1/\mathscr{r}^3)\boldsymbol{\mathscr{r}} = -(1/\mathscr{r}^2)\hat{\boldsymbol{\mathscr{r}}}.$

(c) $\frac{\partial}{\partial x}(\mathscr{r}^n) = n\mathscr{r}^{n-1}\frac{\partial \mathscr{r}}{\partial x} = n\mathscr{r}^{n-1}(\frac{1}{2}\frac{1}{\mathscr{r}}2\mathscr{r}_x) = n\mathscr{r}^{n-1}\hat{\mathscr{r}}_x$, so $\boxed{\nabla(\mathscr{r}^n) = n\mathscr{r}^{n-1}\,\hat{\boldsymbol{\mathscr{r}}}.}$

Problem 1.14

$\overline{y} = +y\cos\phi + z\sin\phi$; multiply by $\sin\phi$: $\overline{y}\sin\phi = +y\sin\phi\cos\phi + z\sin^2\phi$.
$\overline{z} = -y\sin\phi + z\cos\phi$; multiply by $\cos\phi$: $\overline{z}\cos\phi = -y\sin\phi\cos\phi + z\cos^2\phi$.

Add: $\overline{y}\sin\phi + \overline{z}\cos\phi = z(\sin^2\phi + \cos^2\phi) = z$. Likewise, $\overline{y}\cos\phi - \overline{z}\sin\phi = y$.

So $\frac{\partial y}{\partial \overline{y}} = \cos\phi$; $\frac{\partial y}{\partial \overline{z}} = -\sin\phi$; $\frac{\partial z}{\partial \overline{y}} = \sin\phi$; $\frac{\partial z}{\partial \overline{z}} = \cos\phi$. Therefore

$$\left.\begin{aligned} \overline{(\nabla f)}_y &= \frac{\partial f}{\partial \overline{y}} = \frac{\partial f}{\partial y}\frac{\partial y}{\partial \overline{y}} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial \overline{y}} = +\cos\phi(\nabla f)_y + \sin\phi(\nabla f)_z \\ \overline{(\nabla f)}_z &= \frac{\partial f}{\partial \overline{z}} = \frac{\partial f}{\partial y}\frac{\partial y}{\partial \overline{z}} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial \overline{z}} = -\sin\phi(\nabla f)_y + \cos\phi(\nabla f)_z \end{aligned}\right\} \text{ So } \nabla f \text{ transforms as a vector.} \quad \text{qed}$$

Problem 1.16

$$\nabla\cdot\mathbf{v} = \frac{\partial}{\partial x}(\frac{x}{r^3}) + \frac{\partial}{\partial y}(\frac{y}{r^3}) + \frac{\partial}{\partial z}(\frac{z}{r^3}) = \frac{\partial}{\partial x}\left[x(x^2+y^2+z^2)^{-\frac{3}{2}}\right] + \frac{\partial}{\partial y}\left[y(x^2+y^2+z^2)^{-\frac{3}{2}}\right] + \frac{\partial}{\partial z}\left[z(x^2+y^2+z^2)^{-\frac{3}{2}}\right]$$
$$= ()^{-\frac{3}{2}} + x(-3/2)()^{-\frac{5}{2}}2x + ()^{-\frac{3}{2}} + y(-3/2)()^{-\frac{5}{2}}2y + ()^{-\frac{3}{2}} + z(-3/2)()^{-\frac{5}{2}}2z$$
$$= 3r^{-3} - 3r^{-5}(x^2+y^2+z^2) = 3r^{-3} - 3r^{-3} = 0.$$

This conclusion is surprising, because, from the diagram, this vector field is obviously diverging away from the origin. How, then, can $\nabla\cdot\mathbf{v} = 0$? The answer is that $\nabla\cdot\mathbf{v} = 0$ everywhere *except* at the origin, but at the origin our calculation is no good, since $r = 0$, and the expression for $\mathbf{v}$ blows up. In fact, $\nabla\cdot\mathbf{v}$ is *infinite* at that one point, and zero elsewhere, as we shall see in Sect. 1.5.

Problem 1.17

$\overline{v}_y = \cos\phi\, v_y + \sin\phi\, v_z$; $\overline{v}_z = -\sin\phi\, v_y + \cos\phi\, v_z$.

$\frac{\partial \overline{v}_y}{\partial \overline{y}} = \frac{\partial v_y}{\partial \overline{y}}\cos\phi + \frac{\partial v_z}{\partial \overline{y}}\sin\phi = \left(\frac{\partial v_y}{\partial y}\frac{\partial y}{\partial \overline{y}} + \frac{\partial v_y}{\partial z}\frac{\partial z}{\partial \overline{y}}\right)\cos\phi + \left(\frac{\partial v_z}{\partial y}\frac{\partial y}{\partial \overline{y}} + \frac{\partial v_z}{\partial z}\frac{\partial z}{\partial \overline{y}}\right)\sin\phi$. Use result in Prob. 1.14:

$= \left(\frac{\partial v_y}{\partial y}\cos\phi + \frac{\partial v_y}{\partial z}\sin\phi\right)\cos\phi + \left(\frac{\partial v_z}{\partial y}\cos\phi + \frac{\partial v_z}{\partial z}\sin\phi\right)\sin\phi$.

$\frac{\partial \overline{v}_z}{\partial \overline{z}} = -\frac{\partial v_y}{\partial \overline{z}}\sin\phi + \frac{\partial v_z}{\partial \overline{z}}\cos\phi = -\left(\frac{\partial v_y}{\partial y}\frac{\partial y}{\partial \overline{z}} + \frac{\partial v_y}{\partial z}\frac{\partial z}{\partial \overline{z}}\right)\sin\phi + \left(\frac{\partial v_z}{\partial y}\frac{\partial y}{\partial \overline{z}} + \frac{\partial v_z}{\partial z}\frac{\partial z}{\partial \overline{z}}\right)\cos\phi$

$= -\left(-\frac{\partial v_y}{\partial y}\sin\phi + \frac{\partial v_y}{\partial z}\cos\phi\right)\sin\phi + \left(-\frac{\partial v_z}{\partial y}\sin\phi + \frac{\partial v_z}{\partial z}\cos\phi\right)\cos\phi$. So

$$\frac{\partial \overline{v}_y}{\partial \overline{y}} + \frac{\partial \overline{v}_z}{\partial \overline{z}} = \frac{\partial v_y}{\partial y}\cos^2\phi + \frac{\partial v_y}{\partial z}\sin\phi\cos\phi + \frac{\partial v_z}{\partial y}\sin\phi\cos\phi + \frac{\partial v_z}{\partial z}\sin^2\phi + \frac{\partial v_y}{\partial y}\sin^2\phi - \frac{\partial v_y}{\partial z}\sin\phi\cos\phi$$

$$-\frac{\partial v_z}{\partial y}\sin\phi\cos\phi + \frac{\partial v_z}{\partial z}\cos^2\phi$$
$$= \frac{\partial v_y}{\partial y}(\cos^2\phi + \sin^2\phi) + \frac{\partial v_z}{\partial z}(\sin^2\phi + \cos^2\phi) = \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}. \checkmark$$

Problem 1.18

(a) $$\nabla\times\mathbf{v}_a = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & 3xz^2 & -2xz \end{vmatrix} = \hat{\mathbf{x}}(0 - 6xz) + \hat{\mathbf{y}}(0 + 2z) + \hat{\mathbf{z}}(3z^2 - 0) = \boxed{-6xz\,\hat{\mathbf{x}} + 2z\,\hat{\mathbf{y}} + 3z^2\,\hat{\mathbf{z}}.}$$

(b) $$\nabla\times\mathbf{v}_b = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & 2yz & 3xz \end{vmatrix} = \hat{\mathbf{x}}(0 - 2y) + \hat{\mathbf{y}}(0 - 3z) + \hat{\mathbf{z}}(0 - x) = \boxed{-2y\,\hat{\mathbf{x}} - 3z\,\hat{\mathbf{y}} - x\,\hat{\mathbf{z}}.}$$

(c) $$\nabla\times\mathbf{v}_c = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & (2xy + z^2) & 2yz \end{vmatrix} = \hat{\mathbf{x}}(2z - 2z) + \hat{\mathbf{y}}(0 - 0) + \hat{\mathbf{z}}(2y - 2y) = \boxed{0.}$$