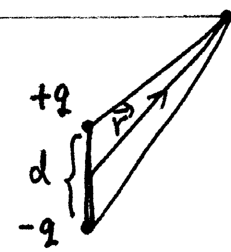
~~Lect 16 Multiple expansion~~



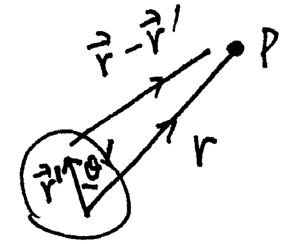
dipole: $$V(r) = q\left(\frac{1}{|\vec{r} - \frac{d}{2}\hat{z}|} - \frac{1}{|\vec{r} + \frac{d}{2}\hat{z}|}\right)$$

$$\frac{1}{|\vec{r} - \frac{d}{2}\hat{z}|} = \frac{1}{\sqrt{r^2 + (\frac{d}{2})^2 - rd\cos\theta}} \approx \frac{1}{r\,(1 - \frac{d}{2r}\cos\theta)} \approx \frac{1}{r}\,(1 + \frac{d}{2r}\cos\theta)$$

$$\frac{1}{|\vec{r} + \frac{d}{2}\hat{z}|} \approx \frac{1}{r}\,(1 - \frac{d}{2r}\cos\theta)$$

$$\Rightarrow V(r) = \frac{q\,d\cos\theta}{r^2}.$$

Generally for a charge distribution $\rho(r)$ in a small region, its potential at the long distance.



$$V(\vec{r}) = \int \frac{\rho(\vec{r}')\,d\tau'}{|\vec{r} - \vec{r}'|}$$

$$\frac{1}{|\vec{r} - \vec{r}'|} = \frac{1}{(r^2 + r'^2 - 2\vec{r}\cdot\vec{r}')^{1/2}} = \frac{1}{r\left[1 + (\frac{r'}{r})^2 - 2\frac{\vec{r}\cdot\vec{r}'}{r^2}\right]^{1/2}}$$

$$= \frac{1}{r}\left[1 + \left(\frac{r'}{r}\right)^2 - \frac{2r'}{r}\cos\theta'\right]^{-1/2}$$

$$= \frac{1}{r}\left[1 + \frac{r'}{r}\cos\theta' + \left(\frac{r'}{r}\right)^2\left[\frac{3\cos^2\theta' - 1}{2}\right] + \left(\frac{r'}{r}\right)^3 \frac{5\cos^3\theta' - 3\cos\theta'}{2} + \dots\right]$$

$$= \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n(\cos\theta')$$

$$\Rightarrow V(r) = \sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\int r'^n P_n(\cos\theta')\,\rho(r')\,d\tau'$$

$$= \underset{\uparrow\ \text{monopole}}{\frac{1}{r}\int \rho(r')\,d\tau'} + \underset{\uparrow\ \text{dipole}}{\frac{1}{r^2}\int r'\cos\theta'\,\rho(r')\,d\tau'} + \underset{\uparrow\ \text{qudrapole}}{\frac{1}{r^3}\int r'^2\left(\frac{3\cos^2\theta' - 1}{2}\right)\rho(r')\,d\tau'}$$

$+\dots$

$$V_{dip}(r) = \frac{1}{r^2}\int r'\cos\theta'\,\rho(\vec{r}')\,d\tau' = \frac{1}{r^2}\int \hat{r}\cdot\vec{r}'\,\rho(\vec{r}')\,d\tau'$$

$$= \frac{1}{r^2}\,\hat{r}\cdot\int \vec{r}'\rho(\vec{r}')d\tau' = \frac{\hat{r}\cdot\vec{p}}{r^2}, \quad \text{where } \underbrace{\vec{p} = \int \vec{r}'\rho(\vec{r}')d\tau'}.$$

$\vec{p}$: diapole momenent.

If we shift the origin $\vec{r}' \to \vec{r}' + \vec{a} \Rightarrow \vec{p}' = \int(\vec{r}'+\vec{a})\,\rho(\vec{r}'+\vec{a})d\tau'$
$= \vec{p} + \vec{a}Q$.

if the total charge $Q=0$, then diapole is indpendent of choice of the origin.

- electric-field of Dipole

$$V = \frac{\vec{p}\cdot\vec{r}}{r^3}, \qquad \vec{E} = -\nabla\cdot V = -\nabla\left(\frac{\vec{p}\cdot\vec{r}}{r^3}\right)$$

$$\nabla\left(\frac{\vec{p}\cdot\vec{r}}{r^3}\right) = \frac{1}{r^3}\nabla(\vec{p}\cdot\vec{r}) - \frac{3(\vec{p}\cdot\vec{r})}{r^4}\,\hat{r}$$

$$\nabla(\vec{p}\cdot\vec{r}) = \vec{p}\times(\nabla\times\vec{r}) + (\vec{p}\cdot\vec{\nabla})\vec{r} = [P_x\partial_x + P_y\partial_y + P_z\partial_z](x, y, z)$$

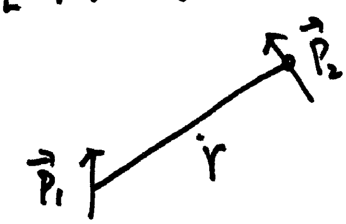
$$= (P_x, P_y, P_z) = \vec{P}$$

$$\Rightarrow \vec{E} = -\frac{\vec{P}}{r^3} + \frac{3(\vec{p}\cdot\hat{r})}{r^3}\,\hat{r} = \frac{1}{r^3}\left[3(\vec{p}\cdot\hat{r})\,\hat{r} - \vec{p}\right]$$

- energy of a dipole in the electric field

$$W = q\left[V(\vec{r}+\vec{d}/2) - V(\vec{r}-\vec{d}/2)\right] = q\,\nabla V\cdot\vec{d} = -\vec{E}\cdot\vec{p}$$

interaction energy between two dipoles



$$W = \frac{1}{r^3}\left[\vec{P}_1\cdot\vec{P}_2 - 3(\vec{P}_1\cdot\hat{r})(\vec{P}_2\cdot\hat{r})\right]$$

set $\vec{P}$ along $\hat{z}$-axis and at the origin

$$\vec{E} = \frac{1}{r^3}\left[\,3\,p\cos\theta\,\hat{e}_r - p\,\hat{z}\,\right] \qquad \hat{z} = \hat{e}_r\cos\theta - \hat{e}_\theta\sin\theta$$

$$= \frac{p}{r^3}\left[\,2\cos\theta\,\hat{e}_r + \sin\theta\,\hat{e}_\theta\,\right]$$