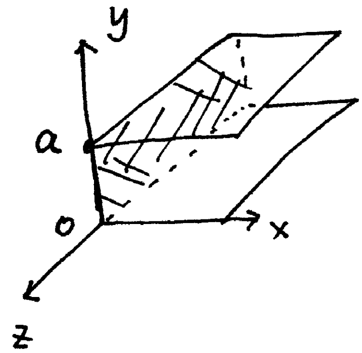
~~Lect 14 Separation of Variables – (I)~~

Cartesian coordinates



example: two infinite grounded metal plates at $y=0$ and $y=a$. The left end at $x=0$ is insulating and maintained at a specfied potential $V_0(y)$. Solve the potential.

The system is uniform along the z-axis $\Rightarrow$ V only depends on (x,y), $V(x,y)$.

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0. \quad \text{Subject to} \quad \begin{cases} V=0 \text{ when } y=0 \text{ and } y=a \\ V = V_0(y) \text{ at } x=0 \\ V \to 0 \text{ as } x \to \infty. \end{cases}$$

search for $V(x,y) = X(x)\,Y(y)$ seperate variable.

$$\Rightarrow \frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} = 0 \Rightarrow \begin{cases} \frac{1}{X}\frac{d^2X}{dx^2} = C \\ \frac{1}{Y}\frac{d^2Y}{dy^2} = -C. \end{cases}$$

"C" can be either positive or negative. In our case, the y-direction $V=0$ at both $y=0$ and a, $\Rightarrow C>0$. rewrite $C=k^2$.

$$\Rightarrow \begin{cases} \frac{d^2X}{dx^2} = k^2 X \\ \frac{d^2Y}{dy^2} = -k^2 Y \end{cases} \Rightarrow \begin{cases} X(x) = A e^{kx} + B e^{-kx} \\ Y(y) = C \sin ky + D \cos ky \end{cases}$$

Because along x-direction, V cannot diverge, thus $A=0$.

$$Y(y=0)=0 \Rightarrow D=0 \Rightarrow V(x,y) = C e^{-kx} \sin ky.$$

$$V(x,a)=0 \Rightarrow k = \frac{n\pi}{a}, \quad n=1,2,\cdots$$

$$\Rightarrow V(x,y) = \sum_{n=1}^{\infty} C_n e^{-n\pi x/a} \sin\left(\frac{n\pi y}{a}\right),$$ C_n: coefficents to be determined.

set $x=0 \Rightarrow V(0,y) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi y}{a}\right) = V_0(y)$.

C_n can be obtained through Fourier transform $\Rightarrow$

$$C_n = \frac{2}{a}\int_0^a V_0(y)\sin\left(\frac{n\pi y}{a}\right) dy.$$

For example if $V_0(y) = V_0 \Rightarrow C_n = \frac{2}{a} V_0 \int_0^a \sin\left(\frac{n\pi y}{a}\right) dy$

$$= \frac{-2V_0}{a} \cdot \frac{a}{n\pi} \cos\left(\frac{n\pi y}{a}\right)\Big|_0^a = \frac{-2V_0}{n\pi}\left[(-)^n - 1\right]$$

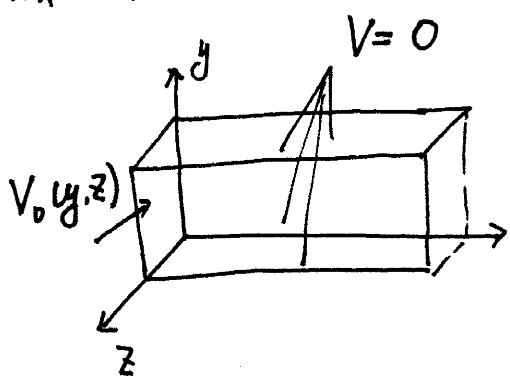
$$= \begin{cases} 0 & \text{if } n \text{ even} \\ \frac{4V_0}{n\pi} & \text{if } n \text{ odd} \end{cases}$$

$$\Rightarrow \boxed{V(x,y) = \frac{4V_0}{\pi} \sum_{n=1,3,5} \frac{1}{n} e^{-\frac{n\pi x}{a}} \times \sin\frac{n\pi y}{a}}$$

Example 3.5

An infinitely long rectangular is grounded, but the left hand at $x=0$, is maintained at $V_0(y,z)$.

Find the potential along the pipe.



$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0,$$ subject to

$V = 0$ for $y = 0, a$
$z = 0, b$

$V \to 0$ for $x \to \infty$

$V = V_0(y,z)$ when $x = 0$.

$$V(x,y,z) = X(x)\,Y(y)\,Z(z) \Rightarrow \frac{1}{X}\frac{d^2X}{dx^2} = C_1,\ \frac{1}{Y}\frac{d^2Y}{dy^2} = C_2,\ \frac{1}{Z}\frac{d^2Z}{dz^2} = C_3$$

with $C_1 + C_2 + C_3 = 0$.

The C_2 and C_3 have to negative, bound from both sides.

$$\Rightarrow \frac{d^2X}{dx^2} = (k_y^2 + k_z^2)\,X, \quad \frac{d^2Y}{dy^2} = -k_y^2\,Y, \quad \frac{d^2Z}{dz^2} = -k_z^2\,Z$$

$$\Rightarrow X(x) = A e^{\sqrt{k_y^2+k_z^2}\,x} + B e^{-\sqrt{k_y^2+k_z^2}\,x}$$

$$\begin{cases} Y(y) = C \sin k_y y + D \cos k_y y \\ Z(z) = E \sin k_z z + F \cos k_z z \end{cases}$$

boundary conditions $\Rightarrow A = 0$, $D = F = 0$

and $k_y = \frac{n_y \pi}{a}$, $k_z = \frac{n_z \pi}{b}$

$$\Rightarrow V(x,y,z) = \sum_{n_y=1}^{\infty}\sum_{n_z=1}^{\infty} C_{n_y,n_z}\, e^{-\pi\sqrt{\left(\frac{n_y}{a}\right)^2 + \left(\frac{n_z}{b}\right)^2}\,x} \sin\frac{n_y \pi y}{a} \sin\frac{n_z \pi z}{b}.$$

Then
$$V(0,y,z) = \sum_{n_y,n_z=1}^{\infty} C_{n_y,n_z} \sin\frac{n_y\pi y}{a} \sin\frac{n_z \pi z}{b} = V_0(y,z)$$

$$\Rightarrow C_{n_y,n_z} = \frac{4}{ab}\int_0^a\int_0^b V_0(y,z) \sin\frac{n_y \pi y}{a} \sin\frac{n_z\pi z}{b}.$$

If $V_0(y,z) = V_0 \Rightarrow C_{n_y,n_z} = \begin{cases} 0 & \text{if } n_y \text{ or } n_z \text{ are even} \\ \frac{16 V_0}{\pi^2 n_y n_z} & \text{both } n_y, n_z \text{ are odd.} \end{cases}$

$$V(x,y,z) = \frac{16 V_0}{\pi^2} \sum_{n_y,n_z=1,3,5}^{\infty} \frac{1}{n_y n_z} e^{-\pi\sqrt{\left(\frac{n_y}{a}\right)^2 + \left(\frac{n_z}{a}\right)^2}} \sin\frac{n_y\pi y}{a} \sin\frac{n_z \pi z}{a}$$

Lect 15 : Separation of variable — spherical coordinate

$$\nabla^2 V=0 \Rightarrow \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial V}{\partial r}\right]+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2}=0$$

for simplicity we only consider the case with azimuthal sym.

$$V(r,\theta)=R(r)\,\Theta(\theta). \Rightarrow$$

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)+\frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)=0$$

$$\left.\begin{aligned}\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)&=\ell(\ell+1)\\ \frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)&=-\ell(\ell+1)\end{aligned}\right\}$$

ℓ has to be integer in order for Θ is regular for $\theta\in[0,\pi]$.

$$\Rightarrow \frac{d}{dr}\left(r^2\frac{dR}{dr}\right)=\ell(\ell+1)R \Rightarrow R(r)=Ar^\ell+B/r^{\ell+1}$$

$$\rightarrow \Theta(\theta)=P_\ell(\cos\theta)$$

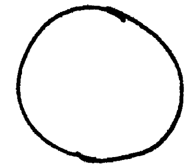
$$P_\ell(x)=\frac{1}{2^\ell\ell!}\left(\frac{d}{dx}\right)^\ell(x^2-1)^\ell$$ — Legendre polynomials.

$$P_0(x)=1,\quad P_1(x)=x,\quad P_2(x)=\frac{1}{2}(3x^2-1)$$

$$\Rightarrow \boxed{V(r,\theta)=\sum_{\ell=0}^{\infty}\left(A_\ell r^\ell+\frac{B_\ell}{r^{\ell+1}}\right)P_\ell(\cos\theta)}$$

example: $V_0(\theta)$ specified on the surface of a hollow sphere

find the potential inside the sphere.



Solution: $V_0(\theta)$ is regular at $r=0 \Rightarrow B_l = 0$

$$V(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$$

set $r=R \Rightarrow V(R,\theta) = \sum_{l=0}^{\infty} A_l R^l P_l(\cos\theta) = V_0(\theta)$

orthogonal relation $\int_{-1}^{1} P_l(x) P_{l'}(x')\, dx = \begin{cases} 0 & l \neq l' \\ \frac{2}{2l+1} & l = l \end{cases}$

$$\Rightarrow A_l R^l \int_0^{\pi} \sin\theta d\theta \, [P_l(\cos\theta)]^2 = \int_0^{\pi} V_0(\theta) P_l(\cos\theta) \sin\theta d\theta$$

$$\Rightarrow A_l R^l \frac{2}{2l+1} = \int \cdots \Rightarrow \boxed{A_l = \frac{2l+1}{2R^l} \int_0^{\pi} \sin\theta d\theta \, V_0(\theta) P_l(\cos\theta)}$$

If we solve the potential outside $\Rightarrow A_l = 0$. (V is regular at $r \to +\infty$)

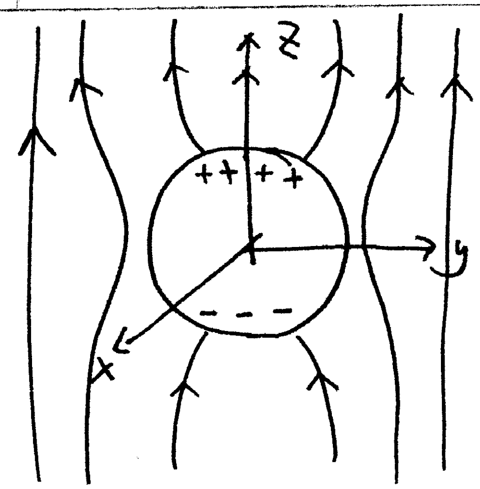
$$V(r,\theta) = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta)$$

set $r=R \Rightarrow V(R,\theta) = \sum_{l=0}^{\infty} \frac{B_l}{R^{l+1}} P_l(\cos\theta) = V_0(\theta)$

$$\Rightarrow \frac{B_l}{R^{l+1}} \frac{2}{2l+1} = \int_0^{\pi} \sin\theta d\theta \, V_0(\theta) P_l(\cos\theta)$$

$$\Rightarrow \boxed{B_l = \frac{2l+1}{2} R^{l+1} \int_0^{\pi} \sin\theta d\theta \, V_0(\theta) P_l(\cos\theta)}$$

ex: An originally uncharged metallic sphere put in an otherwise uniform field $\vec{E} = E_0 \hat{z}$. Solve the potential outside the sphere.



We set the metallic sphere to potential zero.

and at $r \to \infty$, $V \to -E_0 z + \text{const}$

$$\Rightarrow \text{boundary} \begin{cases} V = 0 & \text{at } r = R \\ V \to -E_0 r\cos\theta & \text{for } r \gg R. \end{cases}$$

$$V(r,\theta) = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta) + A_1 r P_1(\cos\theta).$$

as $r \to \infty \Rightarrow A_1 r\cos\theta = -E_0 r\cos\theta \Rightarrow A_1 = -E_0$

$$V(R,\theta) = 0 \Rightarrow \sum_{l=0}^{\infty} \frac{B_l}{R^{l+1}} P_l(\cos\theta) - E R\cos\theta = 0$$

$$\Rightarrow B_l = 0 \text{ for } l \neq 1. \quad B_1 = E R^3 \text{ for } l = 1 \Rightarrow V(r,\theta) = -E_0\left[r - \frac{R^3}{r^2}\right]\cos\theta$$

The extra contribution $V_{induce} = E_0 \frac{R^3}{r^2}\cos\theta$, which is due to a dipole potential.

$$\Rightarrow \sigma(\theta) = -\frac{1}{4\pi}\frac{\partial V}{\partial r}\Big|_{r=R} = -\frac{1}{4\pi}(-)\underbrace{E_0\left[1 + \frac{2R^3}{r^3}\right]}_{\cos\theta}\Big|_{r=R} = \frac{3E_0}{4\pi}\cos\theta$$

electric dipole.

ex: A specified charge density $\sigma_0(\theta)$ on the sphere, find the potential inside and outside.

Solution: inside $r \le R$

$$V_{in}(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta) \quad (r \le R)$$

outside $V_{out}(r,\theta) = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta) \quad (r \ge R)$

$\Rightarrow V_{in}(R,\theta)$ $\Rightarrow A_l R^l = B_l / R^{l+1} \Rightarrow B_l = A_l R^{2l+1}$.

$= V_{out}(R,\theta)$

The surface density $\sigma(\theta) = \frac{-1}{4\pi}\left[\frac{\partial V}{\partial r}\Big|_{out} - \frac{\partial V}{\partial r}\Big|_{in}\right]$

$$\frac{\partial V}{\partial r}\Big|_{out} = \sum_{l=0}^{\infty} \frac{-(l+1)B_l}{r^{l+2}} P_l(\cos\theta)\Big|_{r=R} \qquad \frac{\partial V}{\partial r}\Big|_{in} = \sum_{l=0}^{\infty} A_l\, l\, r^{l-1} P_l(\cos\theta)\Big|_{r=R}$$

$$\Rightarrow \sigma(\theta) = \frac{-1}{4\pi}\left[\sum_{l=0}^{\infty}\left(-\frac{(l+1)B_l}{R^{l+2}} - lA_l R^{l-1}\right)P_l(\cos\theta)\right]$$

$$= \frac{1}{4\pi}\sum_{l=0}^{\infty}(2l+1)A_l R^{l-1} P_l(\cos\theta)$$

$$\Rightarrow \frac{1}{4\pi}\cancel{(2l+1)}A_l R^{l-1} \frac{2}{\cancel{2l+1}} = \int_0^{\pi}\sin\theta\, d\theta\, \sigma(\theta) P_l(\cos\theta)$$

$$\boxed{A_l = \frac{2\pi}{R^{l-1}}\int_0^{\pi}\sigma_0(\theta) P_l(\cos\theta)\sin\theta\, d\theta.}$$

Let us consider a dipole distribution $\sigma_0(\theta) = k\cos\theta$

$$\Rightarrow A_1 = 2\pi k \int_0^{\pi}[P_1(\cos\theta)]^2\, d\cos\theta = \frac{4\pi k}{3}$$

$\Rightarrow$ outside $V(r,\theta) = \frac{4\pi k}{3}\frac{R^3}{r^2}\cos\theta \quad (r>R)$

inside $V(r,\theta) = \frac{4\pi k}{3} r\cos\theta \quad (r<R) \Rightarrow \vec{E}_{in} = -\frac{4\pi k}{3}\hat{z}$.

for the case of last problem, we have $\sigma_0(\theta) = \frac{3E_0}{4\pi} \cos\theta$

i.e $k = \frac{3E_0}{4\pi}$ $\Rightarrow$ The induced electric field

$$\vec{E}_{induce} = -\frac{4\pi}{3} \cdot \frac{3}{4\pi} E_0 \hat{z} = -E_0 \hat{z}$$

which completely cancels the outside field $E_0 \hat{z}$.

and $\vec{E}_{tot} = 0$ inside the metal!