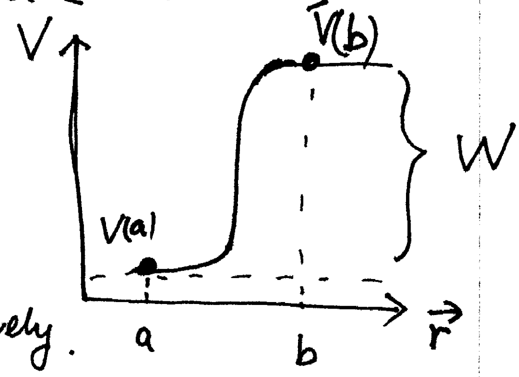
Lect 10 Work and energy

§: work done to move a charge: (against electro-static force

$$W = \int_a^b \vec{F}\cdot d\vec{l} = -Q\int_a^b \vec{E}\cdot d\vec{l} = Q\,[V(b)-V(a)]$$

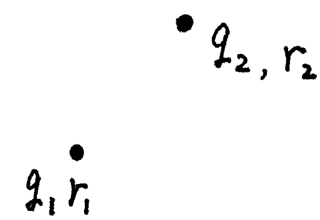


A natural question is that:

Consider a charge distribution

$q_1, \dots q_n$ located at $r_1, \dots r_n$, respectively.

What's the energy stored in such a system?



This energy equals to the work done during the process of bringing the charges from well-separated long distance to the current distribution.

For example, let us consider to charges q_1 and q_2 at r_1 and r_2.

The work to put them togeher: Let us first separate q_1, q_2 at long distance. Then move q_1 to r_1. No work is done during this process. Then q_2 is broght from infinity to r_2. The work done

$$W_{12} = -q_2\int_{+\infty}^{r_2} E\, d\vec{r} = +q_2\,[V(r_2)-V(+\infty)] = \frac{q_2\, q_1}{|r_1-r_2|}$$

- if we have the third charge q_3, and bring it to r_3.

the work done $W = -q_3 \int_{-\infty}^{r_3} (\vec{E}_1 + \vec{E}_2) \cdot d\vec{l} = q_3 \big[V_1(r_3) - V_1(\infty) + V_2(r_3) - V_1(\infty) \big]$

$$= \frac{q_3 q_1}{|\vec{r}_3 - \vec{r}_1|} + \frac{q_3 q_2}{|\vec{r}_3 - \vec{r}_2|}$$

add together, the energy of the three charge system

$$E = \sum W = \frac{q_1 q_2}{|\vec{r}_1 - \vec{r}_2|} + \frac{q_3 q_1}{|\vec{r}_3 - \vec{r}_1|} + \frac{q_3 q_2}{|\vec{r}_3 - \vec{r}_2|}$$

- please check that, if we have n-charge system $(q_1 r_1), (q_2 r_2)$ $\cdots$ (q_n, r_n), then energy is

$$E = \left.\begin{aligned} & \frac{q_2 q_1}{|\vec{r}_1 - \vec{r}_2|} \\ & + q_3 \left(\frac{q_1}{|\vec{r}_3 - \vec{r}_1|} + \frac{q_2}{|\vec{r}_3 - \vec{r}_2|} \right) \\ & + q_4 \left(\frac{q_1}{|\vec{r}_4 - \vec{r}_1|} + \frac{q_2}{|\vec{r}_4 - \vec{r}_2|} + \frac{q_3}{|\vec{r}_4 - \vec{r}_3|} \right) \\ & + \cdots \\ & + q_n \left(\frac{q_1}{|\vec{r}_n - \vec{r}_1|} + \cdots + \frac{q_{n-1}}{|\vec{r}_n - \vec{r}_{n-1}|} \right) \end{aligned}\right\} \begin{aligned} & = \sum_{i<j} \frac{q_i q_j}{|r_i - r_j|} \\ & = \frac{1}{2} \sum_{i \neq j} \frac{q_i q_j}{|r_i - r_j|} \end{aligned}$$

avoid double counting.

or $E = \frac{1}{2} \sum_i q_i \left(\sum_{j \neq i} \frac{q_j}{|r_j - r_i|} \right)$

$= \frac{1}{2} \sum_i q_i V(r_i)$ ← $V(r_i)$ is the potential at r_i generated from all other charges. (not include itself!)

Let consider a continuous charge distribution $\rho(r)$

$$W = \frac{1}{2}\int \rho(r)\, V(r)\, d^3 r,$$

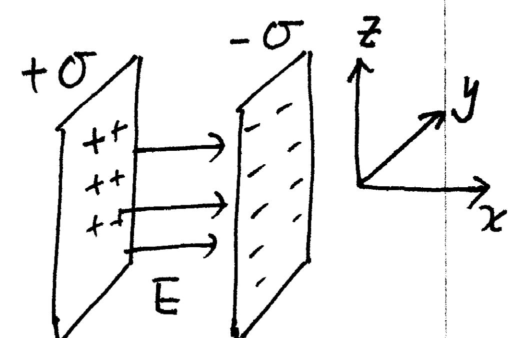
$$= \frac{1}{2}\int \frac{\rho(r)\,\rho(r')}{|\vec{r}-\vec{r}'|}\, d^3\vec{r}\, d^3\vec{r}'$$

for a continuous charge distribution, if $V(r)$ is regular, we don't need to impose $\vec{r}_i \neq \vec{r}'$, as long as the integral converge.

§ Where is the energy stored? — charges or fields

Let us consider two plates with opposite charge density

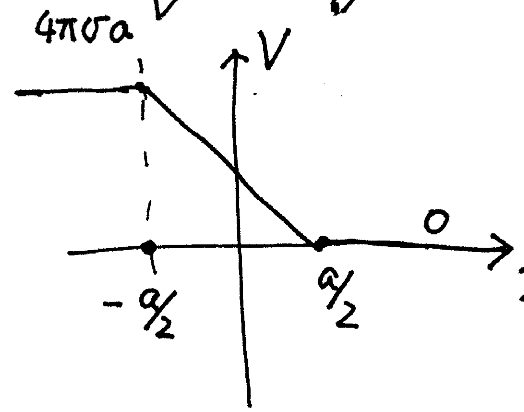
the E field is nonzero between two plates



$$\vec{E} = \begin{cases} 4\pi\sigma\, \hat{x}, & (-\frac{a}{2}<x<\frac{a}{2}) \\ 0 & \text{elsewhere} \end{cases}$$

$\Rightarrow$ the electric potential

$$V(x) = \begin{cases} 0, & x > \frac{a}{2} \\ 2\pi\sigma(a-2x) & -\frac{a}{2}<x<\frac{a}{2} \\ 4\pi\sigma a & x<-\frac{a}{2} \end{cases}$$



$$\Rightarrow W = \frac{1}{2}\int \rho(r)\, V(r)\, d^3 r$$

$$\rho(r) = \sigma\left[\delta(x+a) - \delta(x-a)\right]$$

$$\Rightarrow W = \frac{\sigma}{2}\int dy\, dz \int dx \left[\delta(x+a) - \delta(x-a)\right] V(x,y,z)$$

$$W = \frac{\sigma}{2}\int dy\,dz \left[V\left(\frac{-a}{2}\right) - V\left(\frac{a}{2}\right)\right] = \frac{\sigma}{2}\cdot A \cdot 4\pi\sigma a = 2\pi\sigma^2\,\mathrm{Vol}$$

$$\Rightarrow \boxed{\frac{W}{\mathrm{Vol}} = 2\pi\sigma^2 = \frac{E^2}{8\pi}}$$

~~Suppose~~ that, we enlarge the distance between two plates from $a \to a + da$, what's the work we do? The enenergy density remains $2\pi\sigma^2$, but the volume changes $A\,da$, $\Rightarrow$ the total energy increase!

But what really changed during the process? The charge distribution on two plates doesn't change. It's much more naturally to attribute the energy change is due to the increase of the volume with electric field $\vec{E}$. In other words,

energy is stored in the fields, not just at the location of charges!

We need to change our ideology from charge to field. Charge generates the field, but field gains its own life!

A formal derivation

$$W = \frac{1}{2}\int \rho(r)\, V(r)\, d^3r \,, \quad \leftarrow \quad \nabla\cdot\vec{E} = 4\pi\rho$$

$$= \frac{1}{8\pi}\int \nabla\cdot\vec{E}\; V(r)\, d^3r \quad \leftarrow \text{ partial integral}$$

$$= \frac{1}{8\pi}\left[\int \nabla\cdot(\vec{E}\,V(r))\, d^3r - \int \vec{E}\,\nabla\cdot V(r)\, d^3r\right]$$

$$= \frac{1}{8\pi}\left[\int \vec{E}\cdot\vec{E}\, d^3\vec{r} + \oint_S V\,\vec{E}(r)\cdot d\vec{a}\right]$$

if we integrate over all the space, and set the radius of the surface $\to +\infty$. We know $V \to \frac{1}{R}$, $E \to \frac{1}{R^2}$

the area $\propto R^2$

$\Rightarrow$ the last term $\propto \frac{1}{R} \to 0$.

$\Rightarrow$ $W = \frac{1}{8\pi}\int E^2\, d^3r$ or energy density $\boxed{\rho(r) = \frac{E^2}{8\pi}}$.

Remarks

① What's the energy stored in ~~a point~~ the field, say, electron?

$$E = \frac{e}{r^2} \quad \Rightarrow \quad W = \frac{1}{8\pi}\int_{r_0}^{+\infty} \frac{e^2}{r^4}\cdot 4\pi r^2\, dr$$

$$= \frac{e^2}{8\pi} \cdot 4\pi \int_{r_0}^{+\infty} \frac{dr}{r^2} = \frac{e^2}{2r_0} \to +\infty \quad \text{as } r_0 \to 0.$$

This is indeed a difficulty. Acturally, in order to confine charge in an infinitsimal volume, this kind of diveragence is unavvidable

Acturallby, there's a way to estimate electron radius as

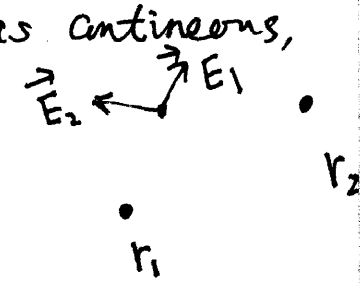
$$\frac{e^2}{r_0} = m_e c^2, \quad r_0 = \frac{m_e c^2}{e^2}, \quad \text{which is called electron em-radians.}$$

$$\sim 2.8 \times 10^{-15}\,\mathrm{m}$$

But actually, we know electron size is much smaller than r_0, if it's not zero.

Nevertheless, this kind of energy divergence of point charge is a frozen energy. It doesn't change, during any process. We only care about the energy difference, thus this is not a problem to us.

If we approximate ~~charge~~ charge distribution as contineous, this difficulty is avoid.



For two charges, the energy stored

$$W_{12} = \frac{1}{8\pi}\left[\int (\vec{E}_1 + \vec{E}_2)^2 dr - \int \vec{E}_1^2 dr - \int \vec{E}_2 dr\right]$$

$$W_{12} = \frac{1}{4\pi} \int \vec{E}_1 \cdot \vec{E}_2 \, d^3r$$

The energy associated with the point charge q_1, q_2 are subtracted

divergent

check $$W_{12} = \frac{-1}{4\pi} \int \vec{E}_1 \; \nabla V_2(r) \, d^3r$$

$$= \frac{-1}{4\pi} \int d^3r \; \nabla(\vec{E}_1 V_2(r_1)) - V_2(r) \nabla \cdot \vec{E}_1 \, d^3r$$

$$= \frac{-1}{4\pi} \left[\oint d\vec{a} \cdot \nabla \left(\vec{E}_1(r) \frac{q_2}{|\vec{r}-\vec{r}_2|}\right)\right] + \frac{1}{4\pi} \int d^3r \frac{q_2}{|\vec{r}-\vec{r}_2|} 4\pi q_1 \delta^3(r-r_1)$$

$$= 0$$

$$\frac{q_1 q_2}{|\vec{r}_1 - \vec{r}_2|}$$

set surface to infinty

$$\left.\begin{array}{l} E \to \frac{1}{R^2} \\ V \to \frac{1}{R} \\ \oint da \to R^2 \end{array}\right\} \frac{1}{R} \to 0$$

$$\Rightarrow \quad W_{12} = \frac{q_1 q_2}{|\vec{r}_1 - \vec{r}_2|} \quad \checkmark$$