Hidden symmetry and exotic quantum magnetism with large-spin alkali and alkaline-earth fermions

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Outline

• Why large spin (multi-component) cold fermions are interesting?

• Large spin ultra-cold fermions are quantum-like NOT semi-classical.

• The simplest case of spin-3/2 fermions are characterized by a generic Sp(4) (SO(5)) symmetry without fine tuning.

• Spin-3/2 Hubbard model unifies antiferromagnetism, superconductivity, and charge-density-wave phases with exact symmetries.

• Exotic “color magnetism” exhibits dominant N-particle correlations (N>=3) --- a feature of QCD.
Why large spin physics with cold atoms is interesting?

- **Novel physics inaccessible** in usual solid state systems.

- **Bosons. spin-1**: $^{23}\text{Na}$, $^{87}\text{Rb}$; spin-2: $^{87}\text{Rb}$; spin-3 $^{52}\text{Cr}$.

  Spinor cond. : Ho and Yip (1998), K. Machida (1998), Ueda, Diener and Ho (2006);

  Topological properties: Zhou (2001--), Demler(2001--), .......

- **Large spin fermions with alkaline-earth and alkali atoms.**

  Fermi liquid and Cooper pairing: Ho and Yip (1999);


  V. Guriare, M. Hermele, A. Rey et al. (2010 ---).
Experiment progress of multi-component fermions

Realization of a $\text{SU}(2) \times \text{SU}(6)$ System of Fermions in a Cold Atomic Gas

Shintaro Taie,¹,* Yosuke Takasu,¹ Seiji Sugawa,¹ Rekishu Yamazaki,¹,² Takuya Tsujimoto,¹ Ryo Murakami,¹ and Yoshiro Takahashi¹,²

Degenerate Fermi Gas of $^{87}\text{Sr}$

B. J. DeSalvo, M. Yan, P. G. Mickelson, Y. N. Martinez de Escobar, and T. C. Killian

Viewpoint

Exotic many-body physics with large-spin Fermi gases

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The experimental realization of quantum degenerate cold Fermi gases with large hyperfine spins opens up a new opportunity for exotic many-body physics.
**Classical (large S):** large-spin solid state systems

- Hund’s rule coupled electrons $\rightarrow$ large onsite spin.
- Inter-site coupling is dominated by exchanging a single pair of electrons.
- $\Delta S_z$ only +1 or -1. Quantum spin-fluctuations are suppressed by $1/S$.

- In solid state systems, the larger the spin is, the more classical the physics is.

- Bilinear exchange dominates

  \[
  \frac{t^2}{U} \mathbf{S}_i \cdot \mathbf{S}_j + \frac{t^3}{U^2} (\mathbf{S}_i \cdot \mathbf{S}_j)^2 + \ldots
  \]

**Not classical but quantum!**: large-spin cold atoms

- Large-spin cold fermion moves as a whole object. The exchange of a pair of fermions can completely flip spin-configuration.

\[ \Delta S_z = \pm 1, \pm 2, \ldots \pm S \]

- Quantum fluctuations are enhanced by the large number of spin components, just opposite to the large-S limit.

- Bilinear, bi-quadratic, bi-cubic terms, etc., are all at equal importance.

\[ \vec{S}_i \cdot \vec{S}_j, (\vec{S}_i \cdot \vec{S}_j)^2, (\vec{S}_i \cdot \vec{S}_j)^3 \]

- Large N instead of large S: SU(2N), Sp(2N); 2N=2S+1.

The simplest case spin-3/2: **Hidden symmetry!**

- **Spin 3/2 atoms**: $^{132}\text{Cs}, ^{9}\text{Be}, ^{135}\text{Ba}, ^{137}\text{Ba}, ^{201}\text{Hg}$. 
  

- **Sp(4) (SO(5))** symmetry without fine tuning regardless of dimensionality, particle density, and lattice geometry!

- **Sp(4)** in spin 3/2 systems $\leftrightarrow$ **SU(2)** in spin 1/2 systems

- **SU(4)** symmetry is realized iff the interaction is spin-independent.

- Importance of high symmetries: unification of competing orders, description of strong spin fluctuations, etc.
Spin-3/2 Hubbard model in optical lattices

\[ H = \sum_{\langle ij \rangle, \alpha} -t \{ c_{i,\alpha}^+ c_{j,\alpha} + \text{h.c.} \} \]  
\[ - \mu \sum_{i} c_{i,\alpha}^+ c_{i,\alpha} \]  
\[ + U_0 \sum_{i} \eta^+(i) \eta(i) + U_2 \sum_{a=1}^{5} \chi_a^+(i) \chi_a(i) \]

- Fermi statistics: only \( F_{\text{tot}} = 0, 2 \) are allowed; \( F_{\text{tot}} = 1, 3 \) are forbidden.

singlet: \[ \eta^+(i) = \sum_{\alpha \beta} \langle 00 | \frac{3}{2} \frac{3}{2} ; \alpha \beta \rangle c_{\alpha}^+(i) c_{\beta}^+(i) \]

quintet: \[ \chi_a^+(i) = \sum_{\alpha \beta} \langle 2a | \frac{3}{2} \frac{3}{2} ; \alpha \beta \rangle c_{\alpha}^+(i) c_{\beta}^+(i) \]

- For arbitrary values of \( t, \mu, U_0, U_2 \) and lattice geometry, there is an \textbf{exact} \( \text{Sp}(4) \), or \( \text{SO}(5) \) symmetry.
What is $\text{Sp}(4)(\text{SO}(5))$ group?

- $\text{SU}(2)$ ($\text{SO}(3))$ group.
  
  3-vector: $x, y, z$; 3-generator: $L_{12}, L_{23}, L_{31}$.
  
  2-spinor: $|\uparrow\rangle, |\downarrow\rangle$

- $\text{Sp}(4)(\text{SO}(5))$ group.

  5-vector: $n_1, n_2, n_3, n_4, n_5$

  **10-generator:** $L_{ab} \ (1 \leq a < b \leq 5)$

  4-spinor: $|\frac{3}{2}\rangle \uparrow, |\frac{1}{2}\rangle \uparrow, |\frac{1}{2}\rangle \downarrow, |\frac{3}{2}\rangle \downarrow$

- For spin-$3/2$ Hubbard model, 3-spin are obviously conserved. What are the other 7-**hidden** conserved quantities?
spin-3/2 algebra $\psi_{\alpha}M_{\alpha\beta}\psi_{\beta}$

- Total degrees of freedom: $4^2 = 16 = 1 + 3 + 5 + 7$.

1 density operator and 3 spin operators are far from complete.

rank: 0 \hspace{1cm} 1, \\
1 \hspace{1cm} $F_x, F_y, F_z$ \\
$M_{\alpha\beta}$ 2 \hspace{1cm} $\xi_{ij}^a F_i F_j (a = 1 \sim 5)$: \\
3 \hspace{1cm} $\xi_{ijk}^a F_i F_j F_k (a = 1 \sim 7)$

- Spin-quadrupole matrices (rank-2 tensors) form five-\(\Gamma\) matrices (SO(5) vector) --- the same \(\Gamma\)-matrices in Dirac equation.

$$\Gamma^a = \xi_{ij}^a F_i F_j, \quad \{\Gamma^a, \Gamma^b\} = 2\delta_{ab}, \quad (1 \leq a, b \leq 5)$$
Hidden conserved quantities: spin-octupoles

• Both $F_{x,y,z}$ and $\xi^a_{ijk} F_i F_j F_k$ commute with Hamiltonian. 10 SO(5) generators: 10=3+7.

• **7 spin-octupole operators** are the hidden conserved quantities.

\[ \Gamma^{ab} = \frac{i}{2} [\Gamma^a , \Gamma^b ] \ (1 \leq a < b \leq 5) \]

• **SO(5): 1 scalar + 5 vectors + 10 generators = 16**

  1 density:
  \[ n = \psi^+ \psi ; \]  
  even

  5 spin-quadrupole: 
  \[ n_a = \frac{1}{2} \psi^+ \Gamma^a \psi ; \]  
  even

  3 spins + 7 spin-octupole:
  \[ L_{ab} = \frac{1}{2} \psi^+ \Gamma^{ab} \psi ; \]  
  odd

1 density:
\[ n = \psi^+ \psi ; \]  
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5 spin-quadrupole:
\[ n_a = \frac{1}{2} \psi^+ \Gamma^a \psi ; \]  
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3 spins + 7 spin-octupole:
\[ L_{ab} = \frac{1}{2} \psi^+ \Gamma^{ab} \psi ; \]  
odd
digression: spin-1/2 Hubbard model (2D square lattice)

- Half-filling – well-known

U>0: spin-SU(2); long range order (LRO) 3-antiferromagnetism (AF)

U<0: pseudo-spin SU(2);
charge density wave (CDW)
+ superconductivity (SC)

--- C. N. Yang’s eta-pairing

- Away from half-filling of U>0 – mostly unknown
Competing orders at 2/4-filling (two particles per site)

- Two types of AF: Sp(4)-adjoint and vector Reps.
  
  A) 10-AF (spin + spin octupole).
  
  B) 5-AF (spin quadrupole).

- Two types of Sp(4) singlet states.
  
  C) SC
  
  D) CDW.

mean-field phase diagram at half-filling (square lattice)
Unifying AF, SC, CDW with even higher exact symmetries!

- **E**: SU(4) line. 15-AF (spin+spin quadrupole+spin octupole).

- **F**: **Exact** SO(7) line. 5-AF + 2-SC=7.

  c.f. SO(5) theory of high Tc: 3-AF + 2 SC=5.

- **G**: Sp(4)*SU(2): CDW+SC; generalization of eta-pairing

- **H**: Exact SO(7) line adjoint Rep. 10-AF + 10-quintet SC + 1-CDW =21 dim
¼-filling (one particle per site) -- “color magnetism”

- Strong spin fluctuations: N=4.
- When the onsite Neel ordering is suppressed, multi-site correlations develop.

- spin-1/2: 2 sites to form an SU(2) singlet.
- 4 sites to form an SU(4) singlet. Each site belongs to the fundamental Rep.

\[
baryon-like \quad \frac{\varepsilon_{\alpha\beta\gamma\delta}}{4!} \psi_1^+ \psi_2^+ \psi_3^+ \psi_4^+ |0\rangle
\]


- c. f. QCD. At least three quarks form an SU(3) color singlet: baryons; multi-particle color/magnetic correlations.
Sp(4) (SO(5)) Heisenberg model at $\frac{1}{4}$-filling

- Spin exchange: bond singlet ($J_0$), quintet ($J_2$). No exchange in the triplet and septet channels.

$$H_{ex} = \sum_{\langle ij \rangle} -J_0 Q_0(ij) - J_2 Q_2(ij)$$

$$J_0 = \frac{4t^2}{U_0}, J_2 = \frac{4t^2}{U_2}, J_1 = J_3 = 0$$

$$\frac{3}{2} \times \frac{3}{2} = 0+2+1+3$$

- Heisenberg model with bi-linear, bi-quadratic, bi-cubic terms.

- SO(5) or Sp(4) explicitly invariant form:

$$H_{ex} = \sum_{ij} \frac{J_0 + J_2}{4} L_{ab}(i)L_{ab}(j) + \frac{-J_0 + 3J_2}{4} n_a(i)n_b(j)$$

$L_{ab}$: 3 spins + 7 spin-octupole tensors; $n_a$: spin-quadrupole operators; $L_{ab}$ and $n_a$ together form the 15 SU(4) generators.
1D lattice (one particle per site)

- Phase diagram is obtained from bosonization analysis and confirmed from DMRG calculations.

- Gapped spin dimer phase at $J_0 > J_2$; bond spin singlet.

- Gapless spin liquid phase at $J_0 \leq J_2$. Spin correlation exhibits 4-site periodicity of oscillations.

Unsolved difficulty: 2D phase diagram

- $J_2 = 0$, Neel ordering obtained by QMC.
  

- $J_2 > 0$, no conclusive results! Difficult both analytically and numerically.

  2D Plaquette ordering at the SU(4) point?

  Exact diagonalization on a 4*4 lattice
  

- Phase transitions as $J_0/J_2$? Dimer phases? Singlet or magnetic dimers?
4x4 Exact diag. (I): Neel correlation

• Spin structure form factor peaks at \((\pi, \pi)\) at \(\theta > 60^\circ\), indicating strong Neel correlation.

\[
S_L(q) \sim \sum_{1 \leq a < b \leq 5} \sum_i \langle L_{ab}(i)L_{ab}(j) \rangle e^{i\mathbf{q} \cdot (\mathbf{r}_i - \mathbf{r}_j)}
\]

4x4 Exact Diag. (II): Dimer correlation

• Susceptibility: \( H(\delta) = H_{\text{exc}} + \delta^* H_{\text{perp}} \) \( E(\delta) = E(0) - \frac{1}{2} \chi \delta^2 \),

• a) Break translational symm:

\[ H_{\text{pert}} = \sum_i \cos(\vec{Q} \cdot \vec{r}_i) H_{\text{ex}}(i, i + x), \]

• b) Break rotational symm:

\[ H_{\text{pert}} = \sum_i H_{\text{ex}}(i, i + x) - H_{\text{ex}}(i, i + y) \]
4x4 Exact Diag. (III): Plaquette formation?

- Local Casimir; analogy to total spin of SU(2).

\[ C(r) \sim \left\langle \sum_{1 \leq a < b \leq 5} \sum_{i \in \text{plaquette } r} L_{ab}(i) \right\rangle^2 \]

\[ C(r) \to 0: \text{ singlet} \]

Open boundary condition
More technical details

HIDDEN SYMMETRY AND QUANTUM PHASES IN SPIN-3/2 COLD ATOMIC SYSTEMS

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Conclusion

• Large-spin cold fermions are quantum-like NOT classical.

• The simplest case of spin-3/2 fermions are characterized by a generic Sp(4) (SO(5)) symmetry without fining tuning.

• Spin-3/2 Hubbard model unifies AF, SC and CDW phases with exact symmetries.

• Exotic “color magnetism” exhibits dominant multi-particle correlations.

Our other work:

• Quintet pairing superfluid and SO(4) Cheshire charge.

• 4-fermion baryon-like superfluidity
Sp(4) (SO(5)) symmetry: the single site problem

$E_0 = 0$

$E_1 = -\mu$

$E_2 = U_2 - 2\mu$

$E_3 = U_0 - 2\mu$

$E_4 = \frac{1}{2}U_0 + \frac{5}{2}U_2 - 3\mu$

$E_5 = U_0 + 5U_2 - 4\mu$

$2^4 = 16$ states.

<table>
<thead>
<tr>
<th></th>
<th>SU(2)</th>
<th>SO(5)</th>
<th>degeneracy</th>
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</thead>
<tbody>
<tr>
<td>$E_{0,3,5}$</td>
<td>singlet</td>
<td>scalar</td>
<td>1</td>
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<tr>
<td>$E_{1,4}$</td>
<td>quartet</td>
<td>spinor</td>
<td>4</td>
</tr>
<tr>
<td>$E_2$</td>
<td>quintet</td>
<td>vector</td>
<td>5</td>
</tr>
</tbody>
</table>

• $U_0 = U_2 = U$, SU(4) symmetry.

$$H_{\text{int}} = \frac{U}{2} n(n-1)$$

• Except 3-spin, what are the 7 **hidden** conserved quantities?
DMRG results in 1D

(a) $\theta = 30^\circ$

(b) $\theta = 60^\circ$
Two-point correlations show four-site periodicity.

\( C(r) \sim \cos(qr) / r^\kappa \)
**Sp(4) magnetism: a four-site problem**

- **Bond spin singlet:**

- **Plaquette SU(4) singlet:**

\[
\frac{\varepsilon_{\alpha\beta\gamma\delta}}{4!} \psi_\alpha^+ \psi_\beta^+ \psi_\gamma^+ \psi_\delta^+ |0\rangle
\]

4-body EPR state; no bond orders

- **Level crossing:**

  d-wave to s-wave

- **Hint to 2D?**
Exact result: SU(4) Majumdar-Ghosh ladder

- Exact dimer ground state in spin 1/2 M-G model.

\[ H = \sum_i H_{i,i+1,i+2}, \quad H_{i,i+1,i+2} = \frac{J}{2} (\vec{S}_i + \vec{S}_{i+1} + \vec{S}_{i+2})^2 \]

- SU(4) M-G: plaquette state.

\[ H = \sum_{\text{every six-site cluster}} H_i \]

\[ H_i = (\sum_{\text{six sites}} L_{ab})^2 + (\sum_{\text{six sites}} n_a)^2 \]

SU(4) Casimir of the six-site cluster

- Excitations as fractionalized domain walls.