Helical Liquid and the Edge of Quantum Spin Hall Systems

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(Received 11 August 2005; published 14 March 2006)

The edge states of the recently proposed quantum spin Hall systems constitute a new symmetry class of one-dimensional liquids dubbed the "helical liquid," where the spin orientation is determined by the direction of electron motion. We prove a no-go theorem which states that a helical liquid with an odd number of components cannot be constructed in a purely 1D lattice system. In a helical liquid with an odd number of components, a uniform gap in the ground state can appear when the time-reversal symmetry is spontaneously broken by interactions. On the other hand, a correlated two-particle backscattering term by an impurity can become relevant while keeping the time-reversal invariance.

DOI: 10.1103/PhysRevLett.96.106401 PACS numbers: 71.55.−i, 72.25.Hg, 73.43.−f, 75.30.Hx

The field of spintronics is motivated largely by the possibility of low power logic devices designed using the spin degree of freedom of the electron [1]. Recently, it has been proposed that, in semiconductor materials with spin-orbit (SO) coupling, a dissipationless spin current can be induced by an electric field [2]. The theoretical prediction of this "intrinsic spin Hall effect" (SHE) [2,3] has stimulated tremendous research activity both theoretical and experimental. On the theoretical side, it has been shown that the vertex correction due to impurity scattering vanishes in the p-doped Luttinger and Rashba models [4,5], while it actually cancels the intrinsic spin Hall effect in the n-doped Rashba model [6]. Recent experimental results in the GaAs system with both electron and hole doping [7,8] are consistent with the existence of a spin Hall effect, although more work is necessary to determine the intrinsic versus extrinsic nature of the observed effect.

However, the electric field in the SHE systems still generates the ordinary Ohmic dissipation in the charge channel of a doped semiconductor. This issue motivated the proposal of a spin Hall insulator [9], where the spin current is not accompanied by the charge current. More recently, the quantum SHE (QSHE) has been proposed in systems with [10] or without Landau levels [11–14]. The QSHE has as a central concept the existence of a bulk gap and gapless edge states in a time-reversal (TR) invariant system with SO coupling. In the ideal QSHE, the left movers on the edge are correlated with down spin $\downarrow$, the right movers have up spin $\uparrow$, and the transport is quantized. We dub the edge states of the QSHE a "helical liquid," which describes the correlation between the spin and the momentum. As spin is not conserved, the extra SO interactions (e.g., Rashba) change the quantized nature of the ideal system. However, the edge transport turns out to be quite robust: As long as the bulk gap is not closed, numerical results find that the spin Hall conductance remains near the quantized value, being rather insensitive to disorder scattering, until the energy gap collapses with increasing SO coupling [15].

The helical liquid constitutes a new symmetry class of 1D liquids; unlike the chiral Luttinger liquid, it does not break TR invariance, and, unlike the usual spinless Luttinger liquid, whose TR transformation satisfies $T^2 = 1$, the TR transformation of the helical liquid satisfies $T^2 = -1$. Unlike the spinful Luttinger liquid, the spinful Luttinger liquid has to have an even number of branches of TR pairs, while the helical liquid can have an odd number of branches of TR pairs. We shall adopt the terminology that an n-component helical liquid contains n TR *pairs* of fermions. Recently, Kane and Mele [16] pointed out that helical liquids with an even or an odd number of components are topologically distinct and are characterized by a Z_2 symmetry in the noninteracting case. This work shows that, in the presence of strong interactions, no strict topological distinctions between the even and odd helical liquids exist, while important quantitative differences do remain.

In this Letter, we analyze the properties of this helical liquid. We write down the lattice Hamiltonian and show that the fermion doubling theorem proves that the helical liquid with an odd number of components cannot be constructed in purely 1D lattices and, hence, must arise as an edge effect of a bulk 2D lattice. In that sense, such a 1D helical liquid must be a "holographic liquid." We then analyze the effects of TR invariant interactions. The umklapp term can open up a gap but at the cost of spontaneously breaking the TR symmetry. Disordered two-particle backscattering also induces a glasslike TR breaking ground state. Fortunately, both cases require extreme repulsive interactions which are unlikely to be experimentally realized.

We begin with the QSHE in the Landau level picture [10], which can be understood as two opposing effective orbital magnetic fields $\pm B$, realized through a special position-dependent SO coupling, acting on spin $\uparrow$ and $\downarrow$ electrons. On a lattice $(x, y) = (m, n)$, with an edge on the x axis, and in the Landau gauge simulated by a linear strain gradient in a sample grown on the $[110]$ direction [10], the Schrödinger equation now becomes

$$E\psi_m(k_y) = -t_x(\psi_{m+1}(k_y) + \psi_{m-1}(k_y)) - \begin{pmatrix} 2t_y\cos(k_y - 2\pi m\phi) & 2\alpha\sin(k_y) \\ 2\alpha\sin(k_y) & 2t_y\cos(k_y + 2\pi m\phi) \end{pmatrix}\psi_m(k_y), \quad (1)$$

with $\psi_m = (\psi_{m,\uparrow}, \psi_{m,\downarrow})^T$ a two-component spinor, flux per plaquette $\phi = p/q$, with p, q relatively prime integers. An extra Rashba SO coupling interaction $-2\alpha k_y\sigma_x$ which mixes the two spins has been added. By using the transfer matrix formalism [17], we can, however, prove that, for $\alpha \ll t_x, t_y$ and to order α, the edge states do not differ in eigenvalues from the $\alpha = 0$ states. There are $q-1$ edge states, and each of them is doubly degenerate. The degeneracy is removed to order α^2.

A generalization of the no-go theorem for chiral fermions in lattice models [18,19] can be used to prove that it is impossible to construct a purely 1D lattice model of the helical liquid with an odd number of components [10,12]. For simplicity, we consider the one-component model with TR symmetry, i.e., two states with orthogonal spin configurations for each momentum k in the Brillouin zone (BZ). The Kramers theorem associated with $T^2 = -1$ ensures that eigenstates with $k = 0$ or π are doubly degenerate. Their energies are denoted by E_0 and E_1, respectively. Without loss of generality, we assume $E_1 > E_0$. TR symmetry also ensures $E_\sigma(k) = E_{\bar{\sigma}}(-k)$ for Kramers doublets with opposite momenta and orthogonal spins. As a result, two dispersion curves start from $k = -\pi$ and converge at $k = 0$ in the left half of the BZ. In the right half of the BZ, the curves are symmetric to those in the left half by a mirror reflection. If an energy E satisfies $E_0 < E < E_1$, it crosses each branch in the left half of the BZ an odd number of times. On the other hand, an energy with $E > E_1$ or $E < E_0$ crosses each branch in the left half of the BZ an even number of times. Thus, for any general energy E, unless it is a local energy maximum or minimum, the total number of crossing points is even in the left half of the BZ as shown in Fig. 1. Thus, the purely 1D band structure generally gives the helical liquid with an even number of components. This result can be straightforwardly generalized to the multiband case. However, the helical liquid with an odd number of components can appear in the edge of a 2D system, because the edge states do not necessarily cover the entire BZ.

TR invariance with $T^2 = -1$ imposes a strong constraint on the correlation function between Kramers pairs. Regardless of interactions and disorder, the Green's function is $G(\vec{r}t\downarrow; \vec{r}t'\uparrow) = \langle\psi_\uparrow(\vec{r}t)\psi_\downarrow^\dagger(\vec{r}'t')\rangle = 0$, where $\langle\rangle$ means thermal average [20]. To generalize this result to multiple pairs, we define a set of fermion annihilation operators of Kramers pairs as $\hat{\psi}_i$ and $\hat{\bar{\psi}}_i$ $(i = 1, 2, \ldots, n)$ which satisfy $T^{-1}\hat{\psi}_i T = \hat{\bar{\psi}}_i$ and $T^{-1}\hat{\bar{\psi}}_i T = -\hat{\psi}_i$. Their $2n$-point correlation functions are defined as

$$G_n(t_1, t_2, \ldots, t_n; t'_n, \ldots, t'_2, t'_1) = \langle\hat{\psi}_1(t_1)\hat{\psi}_2(t_2)\ldots\hat{\psi}_n(t_n)\hat{\bar{\psi}}_n^\dagger(t'_n)\ldots\hat{\bar{\psi}}_2^\dagger(t'_2)\hat{\bar{\psi}}_1^\dagger(t'_1)\rangle. \quad (2)$$

$T^2 = -1$ ensures that $G_n(t_1, t_2, \ldots, t_n; t'_n, \ldots, t'_2, t'_1) = (-)^n G(-t'_1, -t'_2 \ldots - t'_n; -t_n, \ldots - t_2, -t_1)$. Combining this with the time translational symmetry, we obtain

$$G_n(t, t, \ldots, t; 0, 0, \ldots, 0) = 0 \quad (n \text{ is odd}), \quad (3)$$

regardless of interactions and disorder. With the interpretation of ψ and $\bar{\psi}$ as the right and left movers, the observation of Kane and Mele [16] that the single-particle backscattering is forbidden is still correct even in the presence of interactions. However, this does not necessarily mean that the system is gapless. In fact, a two-particle correlated backscattering is allowed as

$$G'_4(t, t; 0, 0) = G_4(t, t; 0, 0) + \langle\hat{\psi}_1(t)\hat{\bar{\psi}}_2^\dagger(0)\rangle\langle\hat{\psi}_2(t)\hat{\bar{\psi}}_1^\dagger(0)\rangle, \quad (4)$$

which effectively describes the propagation of a composite boson and can have nonzero values.

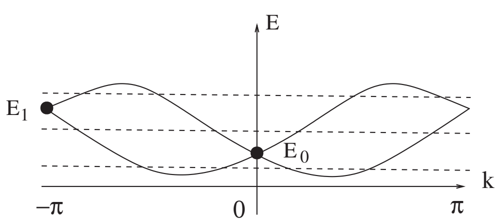

FIG. 1. The band structure in 1D lattice systems with time-reversal symmetry. Even numbers of Kramers doublets appear at a given energy except at the extremum points.

Next we discuss the interaction effects in the one-component model. For simplicity, we consider the band structure with conserved s_z [10,12,13]. After linearizing the spectra around the Fermi points, we arrive at the non-interacting part as

$$H_0 = v_f \int dx(\psi_{R\uparrow}^\dagger i\partial_x\psi_{R\uparrow} - \psi_{L\downarrow}^\dagger i\partial_x\psi_{L\downarrow}), \quad (5)$$

where the right (left) movers $\psi_{R\uparrow}(\psi_{L\downarrow})$ carry spin up (down), respectively, and v_f is the Fermi velocity. A single-particle backscattering term of the form $O_1 = \psi_{R\uparrow}^\dagger\psi_{L\downarrow} + \text{H.c.}$ or $O_2 = i(\psi_{R\uparrow}^\dagger\psi_{L\downarrow} - \text{H.c.})$, which opens up a mass gap in the spinless Luttinger liquid, is not allowed here by virtue of being TR odd. In contrast, for an even number of Kramers pairs $\psi_{iR\uparrow}$, $\psi_{iL\downarrow}$ ($i = 1$–n), it is easy to write down a TR invariant mass term. For $n = 2$, a possible term is $\psi_{1R\uparrow}^\dagger\psi_{2L\downarrow} - \psi_{1R\downarrow}^\dagger\psi_{2R\uparrow} + \text{H.c.}$ The Hamiltonian equation (5) is also different from that of the Luttinger liquid with the Rashba SO coupling [21]. In the latter case, there are still two branches of Kramers pairs.

Only two TR invariant nonchiral interactions are allowed: the forward and umklapp scatterings

$$H_{\rm fw} = g\int dx\psi_{R\uparrow}^\dagger\psi_{R\uparrow}\psi_{L\downarrow}^\dagger\psi_{L\downarrow}, \quad (6)$$

$$H_{\rm um} = g_u \int dx e^{-i4k_f x}\psi^\dagger_{R\uparrow}(x)\psi^\dagger_{R\uparrow}(x+a) \times \psi_{L\downarrow}(x+a)\psi_{L\downarrow}(x) + \text{H.c.}, \quad (7)$$

where a point splitting with the lattice constant a is performed in the umklapp term. The chiral interaction terms only renormalize the Fermi velocity and, thus, are ignored. The umklapp term flips two spins simultaneously, which can be microscopically obtained from anisotropic spin interactions such as $\sum_{\langle ij\rangle} s_x(i)s_x(j) - s_y(i)s_y(j)$ or $\sum_{\langle ij\rangle} s_x(i)s_y(j) + s_y(i)s_x(j)$.

It is well known that the forward scattering term gives the nontrivial Luttinger parameter through $K = \sqrt{(v_f - g)/(v_f + g)}$ but still keeps the system gapless. Only the umklapp term has the potential to open up the gap at commensurate filling $k_f = \pi/2$. The standard bosonized Hamiltonian reads

$$H = \int dx \frac{v}{2}\left\{\frac{1}{K}(\partial_x\phi)^2 + K(\partial_x\theta)^2\right\} + \frac{g_u \cos\sqrt{16\pi}\phi}{2(\pi a)^2}, \quad (8)$$

where $v = \sqrt{v_f^2 - g^2}$ is the renormalized velocity; $\phi = \phi_R + \phi_L$ and $\theta = \phi_R - \phi_L$, respectively. ϕ is also a compact variable with a period of $\sqrt{\pi}$. The standard renormalization group (RG) analysis shows that the umklapp term becomes relevant at $K_c < 1/2$ with a pinned value of ϕ. Consequently, a gap $\Delta \approx a^{-1}(g_u)^{1/2-4K}$ opens, and the spin transport is blocked. The mass order parameters $O_{1,2}$, whose bosonized forms are $O_1 = (i\eta_R\eta_L/2\pi a)\sin\sqrt{4\pi}\phi$ and $O_2 = (i\eta_R\eta_L/2\pi a)\cos\sqrt{4\pi}\phi$, are odd under TR transformation. At $g_u < 0$, ϕ is pinned at either 0 or $\sqrt{\pi}/2$; thus, the O_2 order is Ising-like. At $T = 0$ K, the system is in the Ising-ordered phase; thus, TR symmetry is spontaneously broken. On the other hand, when $0 < T \ll \Delta$, O_2 is disordered in 1D; thus, the gap remains and TR symmetry is restored by thermal fluctuations. Similar reasoning applies to the case of $g_u > 0$ where O_1 is the order parameter.

Now we consider the case that the umklapp term exists only in one single bond. It then behaves as an impurity-induced two-particle correlated backscattering term

$$H'_{\rm bs} = \int dx \delta(x) \frac{g_u}{2(\pi a)^2} \cos\sqrt{16\pi}\phi(x), \quad (9)$$

as depicted in Fig. 2(a). This boundary sine-Gordon (BSG) term was studied in Ref. [22]. It can be reduced to a single-particle problem by integrating out the boson fields ϕ away from $x = 0$, whose effective action for $\phi(x = 0, \tau)$ is equivalent to a 1D classical Coulomb plasma problem [22]. The RG analysis shows that the BSG term becomes relevant at $K < 1/4$. In this case, the 1D line is broken into two separated half-lines; thus, it is insulating for charge transport along the line. However, it remains gapless and can support spin transport. Because the BSG term exists only in a small region, TR symmetry cannot be spontaneously broken. Without loss of generality, we assume $g_u \to -\infty$ in the RG process; then Eq. (9) has two energy minima $\phi(0, \tau) = 0, \sqrt{\pi}/2$, which give $O_2(\tau)$ the same finite amplitude but with opposite signs. As a result, an electron can be backscattered by flipping its spin. The instanton events in Fig. 2(b), i.e., the tunneling processes between these two classical minima, restore TR symmetry. Similar reasoning also applies to the case of $g_u \to +\infty$.

Now we discuss the two-particle backscattering due to quenched disorder, described by the term

$$H_{\rm dis} = \int dx \frac{g_u(x)}{2(\pi a)^2} \cos\sqrt{16\pi}(\phi(x,\tau) + \alpha(x)), \quad (10)$$

where the spatial distribution of scattering strength $g_u(x)$ and phase $\alpha(x)$ are Gaussian random variables. The standard replica analysis shows that the disorder becomes relevant at $K < 3/8$ [23,24]. Then in the ground state $O_{1,2}(x)$ show a glassy behavior, which is disordered in the spatial direction but static in the time direction. Thus, the spin transport is blocked, and TR symmetry is again spontaneously broken at $T = 0$ K. Again at very low but finite T, the system remains gapped with TR symmetry restored.

The conditions for the relevance of two-particle backscattering are $K < 1/2$ for the uniform umklapp scattering and $K < 3/8$ for scattering due to quenched disorder. Since the helical liquid could, *in principle,* open up a gap without breaking the TR symmetry at very low temperatures, we conclude that there is no strict topological distinction [16] between the helical liquid with an even or odd number of components in the presence of disorder and interactions. However, for a reasonably weak interacting system, the one-component helical liquid is expected to remain gapless.

Next we consider the problem of magnetic impurities. The Kondo coupling between the local moment and edge states still keeps the TR invariance. It reads

$$H_K = \int dx \delta(x)\left\{\frac{J_\parallel}{2}(\sigma_- \psi^\dagger_{R\uparrow}\psi_{L\downarrow} + \sigma_+ \psi^\dagger_{L\downarrow}\psi_{R\uparrow}) + J_z\sigma_z(\psi^\dagger_{R\uparrow}\psi_{R\uparrow} - \psi^\dagger_{L\downarrow}\psi_{L\downarrow})\right\}, \quad (11)$$

where $\sigma_\pm = \sigma_x \pm i\sigma_y$, and σ_z denotes the spin 1/2 local moment. $J_\parallel$ describes the spin-flip process accompanied by

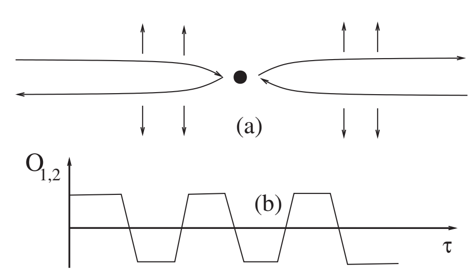

FIG. 2. (a) The two-particle correlated backscattering process at $K < 1/4$ divides the line into two half-lines. (b) The instanton process of $O_{1,2}(0, \tau)$ restores the TR symmetry.

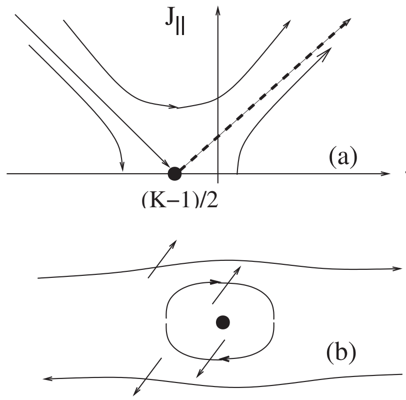

FIG. 3. (a) The RG flow of the Kondo problem in the spin Hall edge state. The antiferromagnetic FP (dashed line) describes the formation of the Kondo singlet. The ferromagnetic FP (solid dot) shows a nonzero critical value of J_z. (b) Kondo screening spin-current vortex around the local moment.

the single-particle backscattering between the left and right channels, while J_z is the nonflipping process with forward scattering of electrons. Combining Eq. (11) and the helical liquid Hamiltonian equations (5) and (7) together, the standard RG analysis [22,25] gives

$$\frac{dJ_z}{d\log L/a} = 2J_{\parallel}^2, \qquad \frac{dJ_{\parallel}}{d\log L/a} = (1 - K + 2J_z)J_{\parallel}, \quad (12)$$

where L is the infrared length scale. Equation (12) can be integrated: $J_{\parallel}^2 - (J_z - (K-1)/2)^2 = c$ (c: constant). A nonzero value of $K-1$ contributes to the RG equation at the tree level due to the anomalous scaling dimension of the backscattering term. Consequently, as shown in Fig. 3(a), the RG flow pattern is shifted as a whole by an amount $(K-1)/2$ along the J_z direction compared to the usual case [25]. The strong coupling antiferromagnetic fixed point (FP) at $J_z \approx J_{\parallel} \to +\infty$ still means the Kondo singlet formation, while the ferromagnetic FP is located at $(J_z, J_{\parallel}) = ((1-K)/2, 0)$, and the Ising coupling cannot induce backscattering. It is remarkable that, with the repulsive forward scattering, i.e., $K < 1$, the Kondo singlet can still form even with a weak ferromagnetic Kondo coupling. After the formation of the Kondo singlet, it behaves like a spinless impurity, which can cause only a phase shift to the edge electrons.

The Kondo singlet in the spin Hall edge exhibits a new feature different from that in the nonchiral systems. According to Nozieres's strong coupling picture [26], an electron from the conduction band is bound around the local moment to form a local singlet. However, the edge electron here has only one helicity. Thus, the spin-flip process inside the Kondo singlet is accompanied with the backscattering of the edge electrons. As a result, the screening electron is actually "circling" around the local moment as depicted in Fig. 3(b). In realistic systems, the width of the edge states is still finite. Thus, the Kondo screening cloud can be considered as a spin-current vortex, and we speculate that its orbital angular momentum is quantized.

In conclusion, we have shown that the edge states of the recently proposed quantum spin Hall systems form a helical liquid, which is a new class different from the spinless or chiral Luttinger liquid. We have shown that this liquid with an odd number of components can arise only as the edge of a 2D system, and we have analyzed its stability under interactions, impurity scattering, and disorder. The new feature of the Kondo singlet was also studied. While there is no strict Z_2 topological distinction between the helical liquids with an even or odd number of components in the presence of strong interactions, the latter is robust in practice against disorder and interactions.

We thank T. L. Hughes and C. L. Kane for helpful discussions. This work was supported by the NSF under Grant No. DMR-0342832 and the U.S. Department of Energy, Office of Basic Energy Sciences under Contract No. DE-AC03-76SF00515. C. W. is also supported by the NSF under Grant No. PHY99-07949.

Note added.—After the completion of our work, we learned that the problem of two-particle scattering in a helical liquid due to quenched disorder has also been independently studied by Xu and Moore [24].

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