Non-Abelian Berry phase and Chern numbers in higher spin-pairing condensates

Chyh-Hong Chern,[1] Han-Dong Chen,[2] Congjun Wu,[1] Jiang-Ping Hu,[3] and Shou-Cheng Zhang[1]
[1]*Department of Physics, McCullough Building, Stanford University, Stanford, California 94305-4045, USA*
[2]*Department of Applied Physics, McCullough Building, Stanford University, Stanford, California 94305-4045, USA*
[3]*Department of Physics and Astronomy, University of California, Los Angeles, California 90095-1547, USA*
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We show that the non-Abelian Berry phase emerges naturally in the *s*-wave and spin quintet pairing channel of spin-3/2 fermions. The topological structure of this pairing condensate is characterized by the second Chern number. This topological structure can be realized in ultracold atomic systems and in solid state systems with at least two Kramers doublets.

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I. INTRODUCTION

Topological gauge structure and Berry's phase[1] play an increasingly important role in condensed matter physics. The quantized Hall conductance can be deeply understood in terms of the first Chern class.[2,3] The fractional quantum Hall effect (FQHE) can be fundamentally described by the $U(1)$ topological Chern-Simons gauge theory.[4] The effective action for ferromagnets[5] and one-dimensional antiferromagnets[6,7] contains Berry phase terms that fundamentally determine the low energy dynamics. More recently, Berry's phase associated with the BCS quasiparticles in pairing condensates has also been studied extensively.[8–10]

While Abelian Berry's phase has found its stage in condensed matter systems, people continue to have an interest in seeking the physical realization of non-Abelian Berry's phase.[11,12] Recently, the non-Abelian SU(2) (Refs. 13–15) Berry's phase (or holonomy, to be precise) has been systematically investigated in the context of condensed matter systems. Demler and Zhang[16] investigated the quasiparticle wave functions in the unified SO(5) theory of antiferromagnetism and superconductivity, and found that the spin density wave (SDW) and the BCS quasiparticle states accumulate an SU(2) Berry's phase (or holonomy) when the order parameter returns to itself after an adiabatic circuit. Zhang and Hu[17] found a higher dimensional generalization of the quantum Hall effect based on a topologically nontrivial SU(2) background gauge field. Rather surprisingly, the non-Abelian SU(2) holonomy also found its deep application in the technologically relevant field of quantum spintronics.[18,19] All these condensed matter applications are underpinned by a common mathematical framework, which naturally generalizes the concept of Berry's $U(1)$ phase factor. This class of applications is topologically characterized by the second Chern class and the second Hopf map, and applies to fermionic systems with time reversal invariance.

In this paper, we investigate the nontrivial topological structures associated with the higher spin condensates. We first review the momentum space gauge structure of the spin-1 condensate, namely the A phase of ^{3}He. As it has been pointed out,[8] the momentum space gauge structure of the pairing condensate is given by that of the t'Hooft-Polyakov monopole. We then investigate the system of spin-2 (quintet) pairing condensate of the underlying spin-3/2 fermions. The most general Hubbard model of spin-3/2 fermions has recently been introduced and investigated extensively by Wu *et al.*,[20] who found that the model always has a generic SO(5) symmetry in the spin sector. Building on this work, we show here that the fermionic quasiparticles of the quintet pairing condensate can be described by the second Hopf map. Similar to the SDW+BCS system investigated by Demler and Zhang,[16] the quasiparticles of the quintet pairing condensate also accumulate an SU(2) holonomy. The quintet pairing condensate can be experimentally realized in a number of systems. Cold atoms with spin-3/2 in the continuum or on the optical lattice can be accurately described by the model of local contact interactions[20] U_0 and U_2. These interaction parameters can be experimentally tuned over a wide range, including the range for stable quintet condensates. Effective spin-3/2 fermions can also be realized in solid state systems with at least two Kramers doublets; for example, in bands formed by $P_{3/2}$ orbitals.

In the rest of this paper, we shall use spin-1/2 system to be short for the spin-1/2 superfluid ^{3}He-A and spin-3/2 system for the *s*-wave spin-3/2 superconductor in the quintet channel. The repeated indices are assumably summed throughout this paper.

II. SUPERFLUID ^{3}He-A

A. Goldstone manifold and the first Hopf map

The general form of the equilibrium order parameter in the ^{3}He-A phase can be written as[8]

$$\langle \Delta(k)_{ai} \rangle = \Delta_k \hat{d}_a (\hat{e}_i^{(1)} + i\hat{e}_i^{(2)}), \tag{1}$$

where the spin index (a) and orbital index (i) run from 1 to 3. Δ_k is a complex number that contains the information of the magnitude and the $U(1)$ phase. $\hat{d}$ is the normal vector of the plane to which the spin direction is restricted. The orthogonal vectors $\hat{e}^{(1)}$, $\hat{e}^{(2)}$, and $\hat{l}=\hat{e}^{(1)}\times\hat{e}^{(2)}$ form a local physical coordinate frame.

The Goldstone manifold of the order parameter is given by[8]

$$R_a = G/H = \frac{U(1) \otimes \mathrm{SO}(3)^{(L)} \otimes \mathrm{SO}(3)^{(S)}}{\mathrm{SO}(2)^{(S)} \otimes U(1)^{combined} \otimes Z_2^{combined}} = S^2 \otimes \mathrm{SO}(3)_{relative}/Z_2. \quad (2)$$

Here, $\mathrm{SO}(3)_{relative}$ denotes such rotations about the axis $\hat{l}$ that lead to new degenerate states that are relative towards gauge transformations.[8] The $U(1)^{combined}$ comes from the fact that the A-phase state is invariant under combined transformation[8] of the gauge transformation with the parameter ϕ from the $U(1)$ group and the orbital rotation of $\hat{e}^{(1)}$, $\hat{e}^{(2)}$, and $\hat{l}$ about axis $\hat{l}$ by the same angle ϕ. The $Z_2^{combined}$ denotes the combined operation that $\hat{d} \to -\hat{d}, \Delta_k \to -\Delta_k$. This combined discrete symmetry leads to the existence of half-quantum vortices.[8] Around a half-quantum vortex, the vector field $\hat{d}$ is continuously rotated into $-\hat{d}$, and the $U(1)$ phase of Δ_k continuously evolves from 0 to π when the order parameter returns to itself after an adiabatic circuit.

If we fix the local orthogonal frame in an arbitrary direction and adiabatically move the quasiparticle around a line defect of a half-quantum vortex, the trajectory of the order parameter is a closed loop on the S^2/Z_2 space. On the other hand, the degrees of freedom of the quasiparticle (a two-dimensional spinor) form a three-dimensional sphere S^3 and the trajectory of the quasiparticle on S^3 is not closed. This adiabatic evolution defines the following map:

$$S^3 \to S^2/Z_2. \quad (3)$$

In the topological terminology, Eq. (3) is determined by the third homopotic group denoted by $\pi_3(S^2/Z_2)$. Due to a theorem in the Homopoty theory,[21]

$$\pi_k(S^n/Z_2) = \pi_k(S^n), \quad \text{for} \quad k \geqslant 2. \quad (4)$$

Equation (3) is homopotically equivalent to the first Hopf map $S^3 \to S^2$, that is $U(1)$ Berry phase in the FQHE and other nanostructures in the semiconductors.[22,23]

B. Berry connection, first Chern number, t'Hooft-Polyakov monopole, and Dirac monopole

If we define the spinor as

$$\Psi_k^\dagger = (c_{k,1/2}^\dagger, c_{k,-1/2}^\dagger, c_{-k,1/2}, c_{-k,-1/2}), \quad (5)$$

the mean field Hamiltonian for ^{3}He-A is given by

$$\mathcal{H} = {\sum_k}' \Psi_k^\dagger H_k \Psi_k, \quad (6)$$

with

$$H_k = \begin{pmatrix} \epsilon_k \sigma^0 & \Delta_k \\ \Delta_k^\dagger & -\epsilon_k \sigma^0 \end{pmatrix}, \quad (7)$$

where $\Delta_k = -\Delta_k \hat{d}_a \sigma^a R$ and $R = -i\sigma^2$. ϵ_k is the kinetic energy on the lattices referenced from the Fermi surface and the summation of momentum k is over half of the Brillouin zone to avoid the double counting. Here, σ^0 is the 2×2 identity matrix and $\sigma^{1,2,3}$ are Pauli matrices.

The Berry phase connection (BPC) is defined by the differential change of states projecting to themselves. In this paper, it will be illustrated by using the state with a positive eigenvalue. BPC obtained from the state with a negative eigenvalue is simply the complex conjugate to the one with the eigenvalue of a different sign. The BPC and its field strength can be obtained, respectively, as

$$A_a = -iA_a^c \frac{\sigma_c}{2}, \quad A_a^c = \epsilon_{abc} \frac{\hat{d}_b}{d}, \quad (8)$$

and

$$F_{bc}^a = \partial_b A_c^a - \partial_b A_c^a + \epsilon_{ade} A_b^d A_c^e = -\frac{1}{d^2} \epsilon_{bce} \hat{d}_e \hat{d}_a. \quad (9)$$

The gauge invariant magnetic field can be defined as

$$B^a = \frac{1}{2} \epsilon_{abc} F_{bc}^e d_e = -\frac{\hat{d}_a}{d^2}. \quad (10)$$

This is a $U(1)$ magnetic-monopole-like field in the d-space. It emerges when there are line defects in the $\hat{d}$-field, e.g., half-quantum vortices in the superfluid He-3A phase. If we transport the spin-1/2 fermion adiabatically around the vortex, the electronic wavefunction gains the phase accumulated due to the $\hat{d}$-field, as we discussed previously. Moreover, the first Chern number can be computed easily, as

$$C_1 = \frac{1}{4\pi} \oint \vec{B} \cdot \vec{dS} = -1. \quad (11)$$

This is the famous t'Hooft-Polyakov monopole (TPM).[8,24] Different from the Dirac monopole, the gauge field of the TPM is non-Abelian and finite everywhere over S^2/Z_2, while the Dirac magnetic monopole is Abelian and has a singularity string. There is a deep and direct relation between them, which can be achieved by a singular gauge transformation.[25–28]

The present SO(3) Berry phase defines a SO(3) gauge theory on S^2/Z_2. Using the covariant derivative $D_a = \partial_a + A_a$, the SO(3) generators in the presence of t'Hooft-Polyakov monopole can be written as[29]

$$L_{ab} = \Lambda_{ab} - id^2 f_{ab}, \quad a = 1,2,3, \quad (12)$$

where $\Lambda_{ab} = -id_a D_b + id_b D_a$ and $f_{ab} = -iF_{ab}^c \sigma_c/2$. Defining $I_a = \frac{1}{2}\epsilon_{abc} L_{bc}$, one finds easily that $[I_a, I_b] = i\epsilon_{abc} I_c$, satisfying the SO(3) algebra. Using Eqs. (8) and (9), one can show

$$L_{ab} = L_{ab}^{(0)} + \epsilon_{abc} \frac{\sigma_c}{2}, \quad (13)$$

where $L^{(0)}$ is the orbital angular momentum, defined by $L_{ab}^{(0)} = -id_a \partial_b + id_b \partial_a$. Define[26]

$$V = \exp\left[i \frac{\vartheta \sigma_{3a} d_a}{\sqrt{d^2 - d_3^2}} \right], \quad (14)$$

where $\sigma_{ab} = \epsilon_{abc} \sigma_c / 2$ and $\cos \vartheta = d_3/d$. One can perform a singular SO(3) gauge transformation on L_{ab} such that $J_{ab} = V L_{ab} V^\dagger$, where

$$J_{\mu\nu} = -id_\mu\partial_\nu + id_\nu\partial_\mu + \epsilon_{\mu\nu}\frac{\sigma_3}{2}, \tag{15}$$

$$J_{\mu 3} = -id_\mu\partial_3 + id_3\partial_\mu - \epsilon_{\mu\nu}\frac{d_\nu}{d+d_3}\frac{\sigma_3}{2}, \tag{16}$$

with $\mu,\nu=1,2$. $\epsilon_{\mu\nu}$ is the antisymmetry tensor, which has only one component: $\epsilon_{12}=1$ in this case. It is obvious that $J_{\mu\nu}=J_{12}$ forms the $U(1)$ generator on S^2/Z_2. From the definition of Eq. (12), one can extract the $U(1)$ BPC regardless of the unnecessary $\sigma_3/2$, as

$$a_\mu = -i\epsilon_{\mu\nu}\frac{d_\nu}{d(d+d_3)}, \quad a_3 = 0, \tag{17}$$

and the finite $U(1)$ field strength over S^2/Z_2 as

$$F_{ab} = i\epsilon_{abc}\frac{d_c}{d^3}. \tag{18}$$

We should notice that the singular gauge transformation we used has a singularity string along the negative z axis. Therefore, while the covariant SO(3) BPC is finite over the whole $\hat{d}$-field, the $U(1)$ BPC has a singularity string that is reflected through the transformation. This transformation is only valid on the northern hemisphere including the equator of the S^2/Z_2. One is able to choose another gauge that has the singularity along the positive z axis to describe the transformation on the southern hemisphere. In the overlap region, the two gauge connections should be connected by a gauge transformation.[30]

The role of this singular gauge transformation is very intriguing. We can view the covariant SO(3) gauge potential A_a^c in Eq. (8) as a vector d_c pointing in the isospin space. The singular gauge transformation is nothing but the rotation of the spin vector from d_c to d_3. Therefore, the invariant subgroup of SO(3) emerges from the isometry group of the equator of S^2/Z_2, which is $U(1)$. This mechanism accounts for the appearance of the $U(1)$ Berry phase in this problem. As a result, the SO(3) Berry phase in this system is essentially equivalent to the $U(1)$ Berry phase.

III. *s*-WAVE QUINTET PAIRING CONDENSATE IN SPIN-3/2 SYSTEM

A. Goldstein manifold and the second Hopf map

Another candidate for nontrivial gauge structures is the spin-3/2 fermionic system with contact interaction, in which an exact SO(5) symmetry was identified recently.[20] It may be studied in the ultracold atomic systems, such as ^{9}Be, ^{132}Cs, ^{135}Ba, ^{137}Ba. The four-component spinor $(c_{3/2},c_{1/2},c_{-1/2},c_{-3/2})^T$ forms the spinor representations of the SU(4) group which is the unitary transformation of the four-component complex spinor. The kinetic energy term has explicit SU(4) symmetry. However, the s-wave contact interaction term breaks the SU(4) symmetry to SO(5). Because of the s-wave scattering, there are only the singlet and quintet channels, as required by the Pauli's exclusion principle. Interestingly, the spin SU(2) singlet and quintet channel interaction can also be interpreted as SO(5) group's singlet and five-vector representations.

The Cooper pair structures have also been studied in spin-3/2 system.[20,31] The singlet and quintet pairing channel operators are described by

$$\eta^\dagger(r) = \frac{1}{2}c_\alpha^\dagger(r)R_{\alpha\beta}c_\beta^\dagger(r),$$

$$\chi_a^\dagger(r) = \frac{-i}{2}c_\alpha^\dagger(r)(\Gamma^a R)_{\alpha\beta}c_\beta^\dagger(r),$$

where Γ^a are the SO(5) gamma matrices that take the form

$$\Gamma^1 = \begin{pmatrix} 0 & iI \\ -iI & 0 \end{pmatrix}, \quad \Gamma^i = \begin{pmatrix} \sigma_i & 0 \\ 0 & -\sigma_i \end{pmatrix}, \quad \Gamma^5 = \begin{pmatrix} 0 & -I \\ -I & 0 \end{pmatrix}, \tag{19}$$

satisfying the Clifford algebra $\{\Gamma^a,\Gamma^b\}=2\delta^{ab}$. The SO(5) charge conjugate matrix R is given by

$$R = \begin{pmatrix} 0 & i\sigma_2 \\ i\sigma_2 & 0 \end{pmatrix}. \tag{20}$$

The quintet pairing structure is spanned by the five polarlike operators $\chi_{1\sim 5}^\dagger$, whose expectation value has a five-vector and a phase structure as $d_a e^{i\phi}$. The Goldstone manifold for the quintet pairing is

$$R_{3/2} = \frac{\mathrm{SO}(5)_s \otimes \mathrm{SO}(3)_L \otimes U(1)}{\mathrm{SO}(4)_s \otimes \mathrm{SO}(3)_L \otimes Z_2} = S^4 \otimes U(1)/Z_2, \tag{21}$$

where the Z_2 symmetry comes from the combined operations $d_a \to -d_a$, $\phi \to \phi + \pi$.

Because the four-component spinor forms the seven-dimensional sphere, similar to the spin-1/2 case, the adiabatic transportation of the quasiparticle around a half-quantum vortex in our spin-3/2 system defines a map

$$S^7 \to S^4/Z_2, \tag{22}$$

which is homopotically equivalent to the second Hopf map, that is, $S^7 \to S^4$.

B. Berry connection, second Chern number, SO(4) monopole, and Yang monopole

Let us introduce the spinor

$$\Psi_k^\dagger = (c_{k,3/2}^\dagger, c_{k,1/2}^\dagger, c_{k,-1/2}^\dagger, c_{k,-3/2}^\dagger, c_{-k,3/2}, c_{-k,1/2}, c_{-k,-1/2}, c_{-k,-3/2}), \tag{23}$$

where $c_{k\sigma}^\dagger$ is the creation operator of an electron with the spin component σ and momentum k. The mean field Hamiltonian can be written as

$$\mathcal{H} = \sum_k{}' \Psi_k^\dagger H_k \Psi_k, \tag{24}$$

with

$$H_k = \begin{pmatrix} \epsilon_k \Gamma^0 & \Delta_k \\ \Delta_k^\dagger & -\epsilon_k \Gamma^0 \end{pmatrix}, \tag{25}$$

where ϵ_k is the kinetic energy on the lattices referenced from the Fermi surface. $\Delta_k = -\Delta_k d_a \Gamma^a R$ while Δ_k contains the magnitude and the phase of the superconducting order parameter. The momentum k is summed only over half of the Brillouin zone to avoid the double counting. The subscript a runs from 1 to 5. d_a forms a four-dimensional sphere S^4. Γ^0 is the 4×4 identity matrix, and Γ^a are given by Eq. (19). The eigenvalues of Eq. (25) are

$$\lambda = \pm E_k = \pm \sqrt{\epsilon_k^2 + |\Delta_k|^2} \tag{26}$$

and their corresponding eigenvectors are

$$\psi_\alpha^+(k) = \frac{1}{\sqrt{(E_k+\epsilon_k)^2 + |\Delta|^2}} \begin{pmatrix} (E_k+\epsilon_k)|\alpha\rangle \\ \boldsymbol{\Delta}_{\mathbf{k}}^\dagger |\alpha\rangle \end{pmatrix}, \tag{27}$$

$$\psi_\alpha^-(k) = \frac{1}{\sqrt{(E_k+\epsilon_k)^2 + |\Delta|^2}} \begin{pmatrix} \boldsymbol{\Delta}_{\mathbf{k}} |\alpha\rangle \\ (E_k+\epsilon_k)|\alpha\rangle \end{pmatrix}, \tag{28}$$

where $|\alpha\rangle$ are SU(4) spinors.

We are interested in the system with the presence of half-quantum vortices. The formation of this kind of vortex is very similar to the ones in the spin-1/2 system. If we transport the spin-3/2 fermion adiabatically around one of them, a nontrivial phase is accumulated due to the $\hat{d}$-field. The BPC and the covariant field strength can be obtained, respectively, by

$$A_a = \frac{i}{d^2} d_c \Gamma^{ca}, \quad a = 1,2,3,4,5 \tag{29}$$

and

$$F_{abc} = -\frac{i}{d^3}(d_a \Gamma^{bc} + d_b \Gamma^{ca} + d_c \Gamma^{ab}), \tag{30}$$

where $\Gamma^{ab} = (1/4i)[\Gamma^a, \Gamma^b]$ making up of the SO(5) generators. Similar non-Abelian gauge structures also appeared in the pseudoparticle field in high dimensions.[32–34] Instead of the first Chern number, we have the nonvanishing second Chern number

$$\mathcal{C}_2 = -\frac{1}{96\pi^2} \oint d\Omega_d \mathrm{Tr}(F_{abc} F_{abc}) = 1, \tag{31}$$

where $d\Omega_d$ denotes the integration over the angular part of d_a. The field strength on S^4, $f_{ab} = [D_a, D_b] = \partial_a A_b - \partial_b A_a + [A_a, A_b]$, can be obtained as

$$f_{ab} = -i \frac{1}{d^2} P_{ab,cd} \Gamma^{cd}, \tag{32}$$

where $P_{ab,cd} = \frac{1}{2}(\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc} + \delta_{ad} d_b d_c - \delta_{bd} d_a d_c - \delta_{ac} d_b d_d + \delta_{bc} d_a d_d)$. $P_{ab,cd}$ is a 10×10 matrix because a and b are antisymmetric as well as c and d. Similar to the projection operator $\delta_{\mu\nu} - \hat{q}_\mu \hat{q}_\nu$ in QED, $P_{ab,cd}$ is the transverse projection operator from ten-dimensional space to six-dimensional space, satisfying $d_a P_{ab,cd} = 0$. The relation between F_{abc} and ordinary field strength f_{ab} can be revealed if we define G_{ab}, which is dual to F_{abc} by $G_{ab} = \epsilon_{abcde} F_{abc}$.[33,34] Then, one can show

$$f_{ab} = \frac{1}{2} \epsilon_{abcde} \frac{d_c}{d} G_{de}. \tag{33}$$

Because of the projection operator $P_{ab,cd}$ in f_{ab}, the fundamental degrees of freedom of the gauge structure in this problem are not SO(5) but SO(4), because SO(4) has six generators. We shall show that it is able to make a route from f_{ab} to the SO(4) gauge field strength using a singular gauge transformation. Similar to the analysis in the second section, the transformation operator in this SO(5) case has the following form:

$$U = \exp\left[i \frac{\vartheta \Gamma^{5\mu} d_\mu}{\sqrt{d^2 - d_5^2}} \right], \quad \mu = 1,2,3,4, \tag{34}$$

where $\cos \vartheta = d_5/d$. Equation (29) then becomes

$$a_\mu = \frac{-i}{d(d+d_5)} \Sigma^{\mu\nu} d_\nu, \quad \mu = 1,2,3,4, \tag{35}$$

$$a_5 = 0, \tag{36}$$

where $\Sigma_{\mu\nu}$ are the SO(4) generators in the $(\frac{1}{2},0)\oplus(0,\frac{1}{2})$ representation, which have the form

$$\Sigma_{\mu\nu} = \begin{pmatrix} \eta^i_{\mu\nu} \frac{\sigma_i}{2} & 0 \\ 0 & \bar{\eta}^i_{\mu\nu} \frac{\sigma_i}{2} \end{pmatrix}, \tag{37}$$

where $\eta^i_{\mu\nu} = \epsilon_{i\mu\nu 4} + \delta_{i\mu}\delta_{4\nu} - \delta_{i\nu}\delta_{4\mu}$ is the t'Hooft symbol, and μ and ν run from 1 to 4. In this reducible representation of the SO(4) gauge group, one can easily distillate the SU(2) ingredients because $SO(4) = SU(2)\otimes SU(2)$. The self-dual SU(2) gauge field is given by

$$a_\mu = \frac{-i}{d(d+d_5)} \eta^i_{\mu\nu} d_\nu \frac{\sigma_i}{2}, \quad \mu = 1,2,3,4,$$

$$a_5 = 0. \tag{38}$$

Similar to the spin-1/2 system, we obtain the SO(4) BPC, which is only defined on the northern hemisphere with the equator. The singularity string along the negative z axis inherits from the singular gauge transformation. The role of the singular gauge transformation by U can be also interpreted as the rotation of a five-dimensional vector from an arbitrary direction d_a to d_5 in the five-dimensional isospin space. Therefore, the invariant subgroup is the isometry group of the equator S^3/Z_3, which is SO(4). Surprisingly, the representation we achieve in SO(4) gauge theory is the reducible $(\frac{1}{2},0)\oplus(0,\frac{1}{2})$, which is the direct sum of two SU(2) gauge theory. Thus, the SU(2) Berry phase naturally arises in this system.

This SU(2) nature of the Berry phase in the spin-3/2 system is manifested if we choose a special spinor $|\alpha\rangle$ such that

$\mathbf{\Delta_k}|\alpha\rangle = |\Delta_\mathbf{k}|e^{-i\phi_\alpha}|\alpha\rangle$, which is studied by Demler and Zhang.[16] In this representation, $|\alpha\rangle$ is not only a SU(4) spinors but also a SO(5) one. The BPC is then given by

$$A_i^{\pm\alpha\beta} = \langle\psi_\alpha^\pm|\partial_i|\psi_\beta^\pm\rangle = \langle\alpha|\partial_i|\beta\rangle. \tag{39}$$

This is exactly the SU(2) holonomy in the context of Demler and Zhang.[16] The special choice of the spinors $|\alpha\rangle$ is equivalent to fixing $\hat{d}_a = \hat{d}_5$ in our notion.

IV. CONCLUSION AND DISCUSSION

In summary, we found that the SU(2) non-Abelian Berry phase emerges naturally in quintet condensates of spin-3/2 fermions. The underlying algebraic structures for the ^{3}He and the spin-3/2 system are the first and the second Hopf maps, respectively. The Chern numbers for both cases were obtained in a standard manner. In the previous case, only the first Chern number is nonvanishing, while in the later case, only second Chern number is nonzero. Both systems appear to have *finite* gauge potential, which means that the BPC can be defined covariantly everywhere over the *d*-field. However, the corresponding $U(1)$ and SU(2) gauge connections can only be defined patch by patch in the *d* space. The bridge across between the finite gauge connection and the one with singularity is constructed by the singular gauge transformation, which can only be defined patch by patch in the *d* space as well.[35] The singular gauge transformation also bear with some physical meaning. It can be understood as the rotation in the spin (isospin) space, namely d_a. When the spin (isospin) points to the north pole, the invariant subgroup becomes the isometry group of the equator. When it is rotated by the gauge transformation, the gauge structure becomes finite and covariant over the whole *d* space.

To experimentally manifest this topological effect, the spin-3/2 ultracold atomic systems may serve as a promising candidate. It may also shed some light on measuring the second Chern number, which has not been revealed by any system so far. Furthermore, our calculation can also be generalized to consider the spin-$\frac{7}{2}$ superconductors in which the algebraic structure is suggested to be the third Hopf map.[36]

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