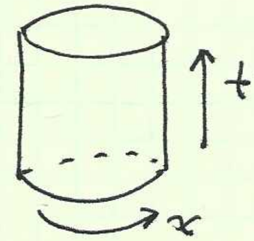
# Lect 5 Free bosons (I)

Consider $\varphi$ a scalar boson field defined on a cylinder satisfying the periodical boundary condition



$$\varphi(t, x) = \varphi(t, x+L).$$

The metric on the cylinder is $g_{\mu\nu} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. (relativistic)

$$\boxed{S[\varphi] = \frac{1}{2g}\int \partial_\mu\varphi\, \partial^\mu \varphi\, dx\, dt = \frac{1}{2g}\int dx dt\; g^{\mu\nu}\partial_\mu\varphi\, \partial_\nu\varphi}$$

where $\partial_\mu\varphi\,\partial^\mu\varphi = -(\partial_t\varphi)^2 + (\partial_x\varphi)^2$.

Action for massless, scalar, free bosonic field (string!).

$g$ is the coupling constant once $\varphi$ is compactified.

**Now Equations of motion:**

$\varphi' = \varphi + \eta$, where $\eta$ is an infinitesimal variation

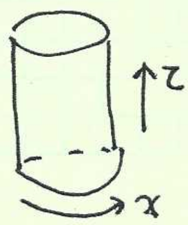
$$\delta S[\varphi] = 1/g \int \partial_\mu\varphi\, \partial^\mu \eta\, dx dt = -1/g \int \eta\, \partial_\mu\partial^\mu\varphi\, dx dt$$

Saddle point solution: Euler-Lagrange Eq $\boxed{\partial_\mu\partial^\mu\varphi = 0.}$

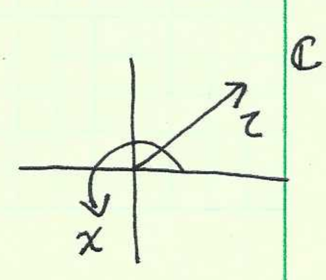
$$\Rightarrow \quad \boxed{\partial_t^2\varphi = \partial_x^2\varphi}$$

In order to show conformal invariance, we perform the Wick rotation

$\tau = -it$



$$z = e^{2\pi(\tau + ix)/L}$$
$$\bar{z} = e^{2\pi(\tau - ix)/L} \rightarrow$$



Euclidean $\Rightarrow S[\varphi] = \frac{1}{g}\int_{\mathbb{C}} \partial\varphi\,\bar\partial\bar\varphi\, dz\, d\bar z$

EOM $\Rightarrow \partial\bar\partial\varphi = 0$, i.e $(\partial_t^2 + \partial_x^2)\varphi = 0 \Rightarrow$ $\begin{cases} \partial\varphi = \partial\varphi(z) \\ \text{holomorphic} \\ \bar\partial\varphi = \bar\partial\varphi(\bar z) \\ \text{anti-holomorphic} \end{cases}$

classical conformal invariance:

The action will not change under

$\begin{cases} z \to z' = z + \varepsilon(z) \\ \bar z \to \bar z' = \bar z + \bar\varepsilon(\bar z) \end{cases}$ and scalar field $\varphi'(z', \bar z') = \varphi(z, \bar z)$.

Proof: $z' = z + \varepsilon(z)$, $\partial_z = \partial_{z'} \frac{\partial z'}{\partial z} = (1 + \partial_z\varepsilon)\,\partial_{z'}$, $dz' = (1 + \partial_z\varepsilon)\, dz$

$$\Rightarrow S[\varphi'] = \frac{1}{g}\int_{\mathbb{C}} \partial'\varphi'(z', \bar z')\, \bar\partial'\varphi'(z, \bar z')\, dz'\, d\bar z'$$

$$= \frac{1}{g}\int_{\mathbb{C}} (1+\partial_z\varepsilon)^{-1}\, \partial\varphi(z,\bar z)\, (1+\partial_{\bar z}\bar\varepsilon)^{-1}\, \bar\partial\varphi(z,\bar z)\, \underbrace{(1+\partial_z\varepsilon)(1+\partial_{\bar z}\bar\varepsilon)}_{dz\,d\bar z}$$

$$= \frac{1}{g}\int_{\mathbb{C}} \partial\varphi(z,\bar z)\, \bar\partial\varphi(z,\bar z)\, dz\, d\bar z$$

$$= S[\varphi].$$

But $S[\varphi]$ is not invariant under a general transformation

$$x'^\mu = x^\mu + \eta^\mu \quad \Rightarrow \quad S[\varphi'] - S[\varphi] = \int T^{\mu\nu}\, \partial_\mu \eta_\nu\, dx\, dt.$$

where $T^{\mu\nu}$ is defined as above. $T^{\mu\nu}$ is the stress-energy tensor.

We take

$z' = z + \eta(z, \bar z)$, $\bar z' = \bar z + \bar\eta(z, \bar z)$, we have

$$S[\varphi'] - S[\varphi] = \int \partial_{z'}\varphi'(z', \bar z')\, \partial_{\bar z'}\varphi'(z', \bar z')\, dz'\, d\bar z' - \int \partial_z\varphi(z,\bar z)\, \partial_{\bar z}\varphi(z,\bar z)\, dz\, d\bar z$$

$$dz' = (1+\partial_z\eta)dz + \partial_{\bar z}\eta\, d\bar z \;, \quad d\bar z' = (1+\partial_{\bar z}\bar\eta)d\bar z + \partial_z\bar\eta\, dz$$

$$\Rightarrow dz' d\bar z' = \begin{vmatrix} 1+\partial_z\eta & \partial_{\bar z}\eta \\ \partial_z\bar\eta & 1+\partial_{\bar z}\bar\eta \end{vmatrix} dz\, d\bar z \approx (1+\partial_z\eta + \partial_{\bar z}\bar\eta)\, dz d\bar z$$

$$\partial_z = \frac{dz'}{dz}\partial_{z'} + \frac{d\bar z'}{dz}\partial_{\bar z'} = (1+\partial_z\eta)\partial_{z'} + \partial_z\bar\eta\, \partial_{\bar z'}$$

$$\partial_{\bar z} = \frac{dz'}{d\bar z}\partial_{z'} + \frac{d\bar z'}{d\bar z}\partial_{\bar z'} = \partial_{\bar z}\eta\, \partial_{z'} + (1+\partial_{\bar z}\bar\eta)\partial_{\bar z'}$$

$$\Rightarrow \begin{cases} \partial_{z'} = (1-\partial_z\eta)\partial_z - \partial_z\bar\eta\, \partial_{\bar z} \\ \partial_{\bar z'} = (1-\partial_{\bar z}\bar\eta)\partial_{\bar z} - \partial_{\bar z}\eta\, \partial_z \end{cases}$$

$$\partial_{z'}\varphi'(z',\bar z') = (1-\partial_z\eta)\partial_z\varphi - \partial_z\bar\eta\, \partial_{\bar z}\varphi$$

$$\partial_{\bar z'}\varphi'(z',\bar z') = (1-\partial_{\bar z}\bar\eta)\partial_{\bar z}\varphi - \partial_{\bar z}\eta\, \partial_z\varphi$$

$\Rightarrow$ Correct to linear order $\Rightarrow$

$$\boxed{\begin{aligned} & S[\varphi'] - S[\varphi] = -1/g \int [\, \partial_z\bar\eta\, \underbrace{\partial_{\bar z}\varphi\, \partial_{\bar z}\varphi} + \partial_{\bar z}\eta\, \underbrace{\partial_z\varphi\, \partial_z\varphi}\,]\, dx\, dt \\ & \Rightarrow \begin{cases} T^{zz} = T^{\bar z\bar z} = 0 \\ T^{z\bar z} = \underbrace{-1/g\, \bar\partial\varphi\, \bar\partial\varphi}_{\text{anti-hol}} \,, \quad T^{\bar z z} = \underbrace{-1/g\, \partial\varphi\, \partial\varphi}_{\text{hol}} \end{cases} \end{aligned}}$$

we often rescal

$$T(z) = 1/2\, \partial\varphi(z)\, \partial\varphi(z),$$

$$\bar T(\bar z) = 1/2\, \bar\partial\varphi(\bar z)\, \bar\partial\varphi(\bar z)$$

$\leftarrow$ at EOM level $\partial\varphi$ is a holomophic function, since $\bar\partial(\partial\varphi) = 0$.

## Canonical quantization:

$$\varphi(t,x) = \sum_{n\in\mathbb{Z}} \varphi_n(t)\, e^{2\pi i n x/L}$$

$\varphi_n(t)$ ← degrees of freedom

$$S = \frac{1}{2g}\int_{-\infty}^{+\infty}\int_0^L -(\partial_t\varphi)^2 + (\partial_x\varphi)^2 \, dx\, dt \quad \leftarrow \quad \mathcal{L} = (\partial_t\varphi)^2 - (\partial_x\varphi)^2$$

$$= -\frac{L}{2g}\int_{-\infty}^{+\infty} dt \sum_{m\in\mathbb{Z}} \left[\dot{\varphi}_m(t)\,\dot{\varphi}_{-m}(t) - \frac{4\pi^2 m^2}{L^2}\varphi_m(t)\,\varphi_{-m}(t)\right]$$

- momentum $\pi_n(t) = \dfrac{\partial\mathcal{L}}{\partial\dot{\varphi}_n(t)} = +\dfrac{L}{g}\dot{\varphi}_{-n}(t)$

- Quantization: treat $\varphi_n$ and $\pi_n$ as operators with

$$\boxed{\begin{aligned} &[\varphi_m(t), \varphi_n(t)] = [\pi_m(t), \pi_n(t)] = 0 \\ &[\varphi_m(t), \pi_n(t)] = i\,\delta_{mn} \end{aligned}} \longrightarrow [\varphi_m(t), \dot{\varphi}_n(t)] = \frac{ig}{L}\delta_{m,-n}.$$

- Complex representation.

$$\partial\varphi(z) = \sum_{n\in\mathbb{Z}} a_n z^{-n-1}, \qquad \bar{\partial}\varphi(\bar{z}) = \sum_{n\in\mathbb{Z}} \bar{a}_n \bar{z}^{-n-1}$$

Then $a_n$, $\bar{a}_n$ are the degree of freedom which become operators in quantum theory.

> Based on Cauchy integral, we have: If $f(z) = \sum_{n\in\mathbb{Z}} f_n z^{-n-1}$,
>
> then $\displaystyle\oint_c f(z)\, z^n \frac{dz}{2\pi i} = f_n$
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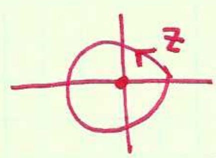
> 
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↑

Residual theorem → Fourier transformation

Generalized theorem:

$$f(z) = \sum_{n\in\mathbb{Z}} f_n (z-w)^{-n-1} \Rightarrow \boxed{\begin{array}{l} \oint_w f(z)(z-w)^n \frac{dz}{2\pi i} = f_n \\ \oint_w \frac{f(z)}{(z-w)^{n+1}} \frac{dz}{2\pi i} = \frac{1}{n!} \partial^n f(w),\ n\in\mathbb{Z} \end{array}}$$

(*) Take the Contour to be a fixed radius, (constant in time)

$$z = e^{2\pi(\tau + ix)/L} \Rightarrow \begin{cases} \tau = \frac{L}{4\pi}(\ln z + \ln \bar{z}) \\ x = \frac{L}{4\pi i}(\ln z - \ln \bar{z}) \end{cases}$$

$$\Rightarrow a_n = \oint_C \partial\varphi(z)\, z^n \frac{dz}{2\pi i}$$

Consider $\frac{\partial}{\partial z} = \frac{\partial x}{\partial z}\cdot\frac{\partial}{\partial x} + \frac{\partial \tau}{\partial z}\cdot\frac{\partial}{\partial \tau}$

$$= \frac{L}{4\pi i}\frac{1}{z}\partial_x + \frac{L}{4\pi z}\partial_\tau$$

$$dz = e^{2\pi(\tau+ix)/L}\frac{2\pi i}{L}dx = \frac{2\pi i}{L} z\, dx$$

for fixed raidus

$$\Rightarrow \partial_z = \frac{L}{4\pi i} z^{-1}(\partial_x + i\partial_\tau) = \frac{L}{4\pi i} z^{-1}(\partial_x - \partial_t) \leftarrow \tau = -it$$

Hence
$$a_n = \frac{1}{4\pi i}\int_0^L (\partial_x\varphi - \partial_t\varphi)\, e^{2\pi i n(x-t)/L}\, dx$$

$$= -\left(\tfrac{n}{2}\varphi_{-n}(t) + \frac{L}{4\pi i}\dot{\varphi}_{-n}(t)\right) e^{-2\pi i n t/L}$$

plug in $\varphi(t,x) = \sum_n \varphi_n(t)\, e^{2\pi i n x/L}$

$$a_n = -\frac{n}{2}\varphi_{-n}(t) - \frac{g}{4\pi i}\pi_n(t)$$

$$\Rightarrow [a_m, a_n] = \frac{m}{2}\frac{g}{4\pi i}[\varphi_{-m}, \pi_n] + \frac{g}{4\pi i}\left(-\frac{n}{2}\right)[\pi_m, \varphi_{-n}]$$

$$= \frac{m}{2}\frac{g}{4\pi}\delta_{m+n,0} + \frac{g}{4\pi}\left(-\frac{n}{2}\right)\delta_{m+n,0}$$

$$= \frac{mg}{4\pi}\delta_{m+n,0}.$$

we set $g = 4\pi$, such that $\boxed{[a_m, a_n] = m\,\delta_{m+n,0}}$

similarly, $\boxed{[a_m, \bar{a}_n] = 0, \quad [\bar{a}_m, \bar{a}_n] = m\,\delta_{m+n,0}}$

## ⊛ Fock space

Given the above algebra of $a_n$ and $\bar{a}_n$, we consider those with $n>0$ as annihilation operators, and those with $n<0$ as creation operators. $a_0$, $\bar{a}_0$ as zero modes.

$a_n^\dagger = a_{-n}$, and $\bar{a}_n^\dagger = \bar{a}_{-n}$, and the zero modes are self-adjoint.

The eigenvalues of $a_0$ and $\bar{a}_0$ are "momenta" in the $z$ and $\bar{z}$-directions. (or the left and right movers).

Consider a vaccum of $a_0, \bar{a}_0$ eigenvalues $|p\bar{p}\rangle$, $p\,\bar{p} \in R$.

We have $\begin{cases} a_n|p\bar{p}\rangle = 0 \\ \bar{a}_n|p\bar{p}\rangle = 0 \end{cases}$ with $n>0$.

and $a_0|p\bar{p}\rangle = p|p\bar{p}\rangle$, $\bar{a}_0|p\bar{p}\rangle = \bar{p}|p\bar{p}\rangle$.

Then the states built based on applying $a_n$ and $\bar{a}_n$ with $n<0$, are excited states of the vaccum $|p\bar{p}\rangle$. — Fock space $F_{p\bar{p}}$.

For example $a_{-1}|p\bar{p}\rangle$, $\bar{a}_{-2}|p\bar{p}\rangle$, $a_{-3}^2\,\bar{a}_7|p\bar{p}\rangle$, etc.

Since $[a_0, a_n] = 0$, and $[\bar{a}_0, \bar{a}_n] = 0$, the states in the Fock space $F_{p\bar{p}}$ share the same eigenvalues of $a_0$ and $\bar{a}_0$. For example $a_0(a_{-1}|p\bar{p}\rangle)$
$= a_{-1}a_0|p\bar{p}\rangle = p(a_{-1}|p\bar{p}\rangle)$.

But they have more energy. To see this, we need to use the stress-energy tensor $T(z)$, $\bar{T}(\bar{z})$.

Consider $T(z) = \frac{1}{g}\, \partial_z \varphi\, \partial_z \varphi \leftarrow \partial_z \varphi(z) = \sum_{n \in \mathbb{Z}} a_n z^{-n-1}$

$$= \frac{1}{g} \sum_{r,s} a_r a_s z^{-r-s-2}$$

$$= \frac{1}{g} \sum_{n \in \mathbb{Z}} \Big[ \sum_{r \in \mathbb{Z}} a_r a_{n-r} \Big] z^{-n-2}$$

Define $L_n \equiv \sum_{r \in \mathbb{Z}} a_r a_{n-r}$. We need to define normal ordering to avoid diverages.

Check $L_0 |p\rangle = \frac{1}{g} \sum_{r \in \mathbb{Z}} a_r a_{-r} |p\rangle = \frac{1}{g} \Big[ \cancel{\sum_{r=-\infty}^{-1} a_r a_{-r} |p\rangle} + a_0^2 |p\rangle + \sum_{r=1}^{\infty} a_r a_{-r} |p\rangle \Big]$

$$= \frac{1}{g} p^2 |p\rangle + \frac{1}{g} \sum_{r=1}^{\infty} \left( a_{-r} a_r + [a_r\ a_{-r}] \right) |p\rangle$$

$$= \frac{1}{g} \Big[ p^2 + \underbrace{\sum_{r=1}^{\infty} r}_{\text{diverges!}} \Big] |p\rangle$$

Hence we define normal ordering $:a_m a_n: = \begin{cases} a_m a_n & (m \le -1,\ a_m \text{ is a creater}) \\ a_n a_m & \text{if } m \ge 0 \ (a_m \text{ is not a creater}) \end{cases}$

Quantum stress-energy tensor:

$$T(z) = \frac{1}{g} : \partial\varphi(z) \partial\varphi(z) : = \frac{1}{g} \sum_{n \in \mathbb{Z}} \Big[ \underbrace{\sum_{r \in \mathbb{Z}} : a_r a_{n-r} :}_{L_n} \Big] z^{-n-2}$$

$$= \frac{1}{g} \sum_{n \in \mathbb{Z}} \Big[ \sum_{r \le -1} a_r a_{n-r} + \sum_{r \ge 0} a_{n-r} a_r \Big] z^{-n-2}$$

$$T(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2}$$

Then $L_n$ act on an excited state without diverages.

For example: $$L_0|p\rangle = \frac{1}{g}\left[\sum_{r\le -1} a_r a_{-r}|p\rangle + \sum_{r\ge 0} a_{-r}a_r|p\rangle\right]$$
$$= \tfrac{1}{g}\, a_0^2|p\rangle = \frac{1}{g}\, p^2|p\rangle.$$

The sum of the eigenvalues of $L_0$ and $\overline{L}_0$ is called the energy.

※ Please check

$[L_m, a_n] = -n\, a_{m+n}$, for all $m, n, \in \mathbb{Z}$

$$\boxed{[L_m, L_n] = (m-n) L_{m+n} + \frac{m^3-m}{12}\delta_{m+n,0}\, c, \quad \text{here } c=1.}$$

Compare to the classical case

$[l_m, l_n] = (m-n)\, l_{m+n}$, there's an extra term due to quantization. $C$ is called the central charge.

For the anti-holomorphic part.

$$\overline{T}(\overline{z}) = \sum_{n\in\mathbb{Z}} \overline{L}_n \overline{z}^{-n-2}, \quad \overline{L}_n = \tfrac{1}{g}\sum_{r\in\mathbb{Z}} :\overline{a}_r\overline{a}_{n-r}:, \quad \overline{c}=1,$$

so, we have $C = \overline{C} = 1$.

Since $[L_0, a_{-n}] = n\, a_{-n}$, it means that $a_{-n}$ is an eigen-operator such that acting a creation operator $a_{-n}$ increases energy by $n$ units.

$$L_0|\psi\rangle = E|\psi\rangle \Rightarrow L_0 a_{-n}|\psi\rangle = (a_{-n}L_0 + [L_0, a_{-n}])|\psi\rangle$$
$$= (E+n)\, a_{-n}|\psi\rangle.$$

Here is the summary of what we have

Canonical quantization: $\partial\varphi(z) = \sum_{n\in\mathbb{Z}} a_n z^{-n-1}$

$$[a_m, a_n] = m\,\delta_{m+n,0}$$

Fock spaces: $|p\rangle$ vacuum of the holomophic sector
$|\bar{p}\rangle$ . . . . . . . . anti-holomorphic sector

$a_0|p\rangle = p|p\rangle$, $a_n|p\rangle = 0$, $\forall n>0$

$\bar{a}_0|\bar{p}\rangle = \bar{p}|\bar{p}\rangle$, $\bar{a}_n|\bar{p}\rangle = 0$,

$a_{-n}$ and $\bar{a}_{-n}$ act on $|p\rangle$ ($|\bar{p}\rangle$) to give excited states in the same conformal tower.

$a_{-1}^3|p\rangle$, $a_{-2}a_{-1}|p\rangle$, $a_{-3}|p\rangle$

$a_{-1}^2|p\rangle$ $a_{-2}|p\rangle$ $E/g = \frac{1}{2}p^2+2$

$a_{-1}|p\rangle$ $E/g = \frac{1}{2}p^2+1$

$|p\rangle$ $\frac{E}{g} = \frac{1}{2}p^2$

normal ordering

$$T(z) = \frac{g}{2}\left(\partial\varphi(z)\right)^2 = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$$

$$:a_m a_n: = \begin{cases} a_m a_n & \text{if } m \le -1 \\ a_n a_m & \text{if } m \ge 0 \end{cases}$$

$\Rightarrow L_n = \frac{g}{2}\sum_{r\in\mathbb{Z}} :a_r a_{n-r}:$ and

$$T(z) = \frac{g}{2} :\partial\varphi(z)\,\partial\varphi(z): = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$$

$a_0$ & $\bar{a}_0$ : momentum / particle # operator

$L_0 + \bar{L}_0$ = energy operator, $L_0 - \bar{L}_0$ = angular momentum operator

$[L_m, a_n] = -n\, a_{m+n}$,

$\boxed{c=1}$

$$\boxed{[L_m, L_n] = (m-n)L_{m+n} + \frac{c\,m(m+1)(m-1)}{12}\delta_{m+n,0}}$$

Virasoro algebra

Free boson is a quantum conformal field theory.

Check commutator $[L_m, a_n] = -n\, a_{m+n}$, $[L_m, L_n] = (m-n) L_{m+n} + \frac{m(m^2-1)}{12} \delta_{m+n,0}\, c$

$$L_n = \frac{1}{2} : \sum_{r\in\mathbb{Z}} a_r a_{n-r} : = \begin{cases} \frac{1}{2} \sum_{r\in\mathbb{Z}} a_r a_{n-r} & (n\neq 0) \\ \sum_{r>0} a_{-r} a_r + \frac{1}{2} a_0^2 & (n=0) \end{cases}$$

$[a_m, a_n] = m\, \delta_{m+n,0}$

Check: ① $m \neq 0$, $[L_m, a_n] = \frac{1}{2} \sum_r \{ a_r [a_{m-r}, a_n] + [a_r, a_n] a_{m-r} \}$

$$= \frac{1}{2} \sum_r r\, \delta_{r,-n}\, a_{m-r} + (-n)\, \underbrace{\delta_{m-r,-n}}_{a_r} = -n\, a_{m+n}$$

$m = 0$, $[L_0, a_n] = \sum_{r>0} [a_{-r} a_r, a_n] + \frac{1}{2} [a_0^2, a_n]$

$$= \sum_{r>0} a_{-r} [a_r, a_n] + [a_{-r}, a_n] a_r + \frac{1}{2} \{ [a_0, a_n] a_0 + a_0 [a_0, a_n] \}$$

$$= \sum_{r>0} a_{-r}\, r\, \delta_{r,-n} + \underbrace{\phantom{x}}_{-r} \delta_{n,r}\, a_r = \begin{cases} -n\, a_n & n>0 \\ -n\, a_n & n\leq 0 \end{cases}$$

$$= -n\, a_n$$

$\Rightarrow [L_m, a_n] = -n\, a_{m+n}$

② $[L_m, L_n]$ : if $n \neq 0$, then $L_n = \frac{1}{2}\sum_{r\in\mathbb{Z}} a_r a_{n-r}$,

$$[L_m, L_n] = \frac{1}{2} \sum_{r\in\mathbb{Z}} [L_m, a_r a_{n-r}] = \frac{1}{2} \sum_{r\in\mathbb{Z}} [L_m, a_r] a_{n-r} + a_r [L_m, a_{n-r}]$$

$$= \frac{1}{2} \sum_{r\in\mathbb{Z}} \left[ (-r)\, a_{m+r}\, a_{n-r} - (n-r)\, a_r\, a_{m+n-r} \right]$$

if $m+n \neq 0$, each term of the above expression is already normal ordering

it becomes $\frac{1}{2}\left[ \sum_{r'\in\mathbb{Z}} \underbrace{a_{r'} a_{m+n-r'}}_{(m-r')} - \sum_{r\in\mathbb{Z}} (n-r)\, a_r\, a_{m+n-r} \right]$ ← shift $r' = m+r$

$$= \frac{1}{2} \sum_{r\in\mathbb{Z}} (m-n)\, a_r\, a_{m+n-r} = (m-n) L_{m+n}$$

if $m+n=0$, $\Rightarrow$ Let us assume $m>0$ and $n<0$

$$[L_m, L_n] = \frac{1}{2}\sum_{r\in\mathbb{Z}} (-r)\, a_{m+r}\, a_{-m-r} - (n-r)\, a_r\, a_{-r}$$

$$\sum_{r\in\mathbb{Z}} (-r)\, a_{m+r}\, a_{-m-r} = \sum_{r<-m} (-r)\, a_{m+r}\, a_{-m-r} + m a_0^2 + \sum_{r>-m} (-r)\, a_{m+r}\, a_{-m-r}$$

$$= \sum_{r<-m} (-r)\, a_{m+r}\, a_{-m-r} + m a_0^2 + \sum_{r>-m} (-r)\left[ a_{-m-r}\, a_{m+r} + (m+r)\right]$$

$$= \sum_{r'<0} (m-r')\, a_{r'} a_{-r'} + m a_0^2 + \sum_{r'>0} (m-r')\, a_{-r'} a_{r'} + \sum_{r>-m} (-r)(m+r)$$

$$\sum_{r\in\mathbb{Z}} (n-r)\, a_r a_{-r} = \sum_{r<0} (n-r)\, a_r a_{-r} + n a_0^2 + \sum_{r>0} (n-r)\left( a_{-r} a_r + r\right)$$

$$= \sum_{r<0} (n-r)\, a_r a_{-r} + n a_0^2 + \boxed{\sum_{r>0} (n-r)(a_{-r} a_r) + \sum_{r>0} (-m-r)\, r}$$

$$\Rightarrow [L_m, L_n] = \tfrac{1}{2}(m-n)\left[ \sum_{r<0} a_r a_{-r} + a_0^2 + \sum_{r>0} a_{-r} a_r \right]$$

$$+ \left[\sum_{r>-m} (-r)(m+r) - \sum_{r>0} (-m-r)\, r\right] / 2$$

$$= (m-n) L_0 + \frac{1}{2}\sum_{r=-m+1}^{0} (-r)(m+r) \leftarrow \sum_{r=0}^{m-1} r(m-r)$$

$$= \sum_{r=1}^{m-1} r(m-r)$$

$$= 1\cdot m-1 + 2\cdot(m-2) + \cdots (m-1)\cdot 1$$

$$= m\frac{(m-1)m}{2} - \frac{(m-1)m(2m-1)}{6}$$

$$= (m-1)m\left[\frac{m}{2} - \frac{2m-1}{6}\right]$$

$$= \frac{(m-1)m(m+1)}{6}$$

$$= (m-n) L_0 + \frac{1}{12} m(m^2-1).$$

$\boxed{\text{if } n=0, \text{ it can also be proved similarly.}}$