Conformal Ward-identity

Warm up

$$Z = z^0 + iz^1 \qquad z^0 = \frac{1}{2}(z+\bar z) \qquad \partial_z = \frac{1}{2}(\partial_0 - i\partial_1) \qquad \partial_0 = \partial_z + \partial_{\bar z}$$

$$\bar Z = z^0 - iz^1 \qquad z^1 = \frac{1}{2i}(z-\bar z) \qquad \partial_{\bar z} = \frac{1}{2}(\partial_0 + i\partial_1) \qquad \partial_1 = i(\partial_z - \partial_{\bar z})$$

$$\boxed{\begin{array}{l} z^0 = \tau \\ z^1 = x \end{array}}$$

$$Z^\mu = \begin{pmatrix} z \\ \bar z \end{pmatrix} \qquad z_\mu = g_{\mu\nu} z^\nu = \left(\frac{\bar z}{2}, \frac{z}{2}\right)$$

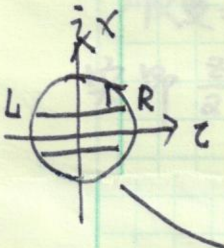
$$ds^2 = g_{\mu\nu} dZ^\mu dz^\nu = dz\, d\bar z \Rightarrow g_{\mu\nu} = \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}$$

$$= g^{\mu\nu} dz_\mu dz_\nu \qquad g^{\mu\nu} = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$$

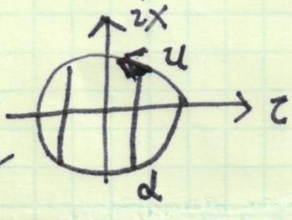
我们用的还是欧氏空间度规

$ds^2 = dz\,d\bar z$
$= (dx)^2 + (d\tau)^2$

Integrals ① $\int_M dx^2\, \partial_{\bar z} f(z) = \int d\tau\, dx\, \frac{1}{2}(\partial_\tau + i\partial_x) f(z)$



$$= \int \frac{dx}{2} \int d\tau\, \partial_\tau f(z) + i \int \frac{d\tau}{2} \int dx\, \partial_x f(z)$$



$$= \int \frac{dx}{2}\; f_R(z) - f_L(z) + i \int \frac{d\tau}{2}\; f_u(z) - f_d(z)$$

$$= -\frac{i}{2}\left[\int_{\partial M} d\tau\; f_d(z) - f_u(z) + i \int_{\partial M} dx\; f_R(z) - f_L(z)\right]$$

$$= -\frac{i}{2} \oint_{\partial M,\,\text{反时针}} dz\; f(z) \quad \leftarrow \text{check}$$

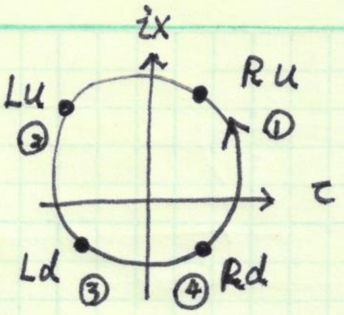
$$\Rightarrow \boxed{\int_M d^2x\, \partial_{\bar z} f(z) = \frac{-i}{2} \oint_{\partial M} dz\; f(z)}$$

同理

$$\boxed{\int_M d^2x\, \partial_z f(\bar z) = \frac{i}{2} \oint_{\partial M} d\bar z\; f(\bar z)}$$

*) comment

$$\oint dz\, f(z) = \int^{①} -d\tau + i\,dx\; f_{R,u}(z) + \int^{②} -d\tau - i\,dx\; f_{L,u}(z) + \int^{③} d\tau - i\,dx\; f_{L,d} + \int^{④} d\tau + i\,dx\; f_{R,d}$$



这里 we set $d\tau > 0$, $dx > 0$.

$$= \underbrace{\int d\tau\,(f_{Ld} - f_{Lu})}_{左半} + \underbrace{\int d\tau\,(f_{Rd} - f_{Ld})}_{右半}$$

不论 f

$$+ \underbrace{i\int dx\,(f_{Ru} - f_{Lu})}_{上半} + \underbrace{i\int dx\,(f_{Rd} - f_{Ld})}_{下半}$$

$\boxed{\text{确为逆时针回路!}}$

$$= \int_{从左\to右} d\tau\; f_d - f_u + i\int_{从下\to上} dx\; f_R - f_L$$

② Prove $\boxed{\begin{aligned}\delta^{(2)}(x) &= \frac{1}{\pi}\partial_{\bar z}\left(\frac{1}{z}\right)\\ &= \frac{1}{\pi}\partial_z\left(\frac{1}{\bar z}\right)\end{aligned}}$ Please note that this is the 2D δ-function.

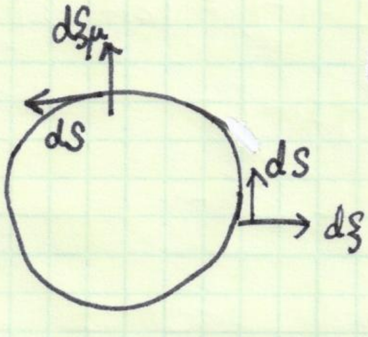
$$\frac{1}{\pi}\iint_M d^2x\; \partial_{\bar z}\left(\frac{f(z)}{z}\right) = \frac{1}{\pi}\iint_M d^2x\, f(z)\,\partial_{\bar z}\left(\frac{1}{z}\right)$$

$$\downarrow\; \frac{1}{2\pi i}\oint_{\partial M} dz\,\frac{f(z)}{z} = f(0) = \int_M d^2x\;\delta^2(x)\,f(z)$$

$$\frac{1}{\pi}\iint_M d^2x\;\partial_z\left(\frac{f(\bar z)}{\bar z}\right) = \frac{1}{\pi}\iint_M d^2x\, f(\bar z)\,\partial_z\left(\frac{1}{\bar z}\right)$$

$$\downarrow\; \frac{-1}{2\pi i}\oint_{\partial M} d\bar z\,\frac{f(\bar z)}{\bar z} = f(0) = \int_M d^2x\;\delta^2(x)\,f(z)$$

$$\int_M d^2x\, \partial_\mu F^\mu = \int_{\partial M} d\xi_\mu F^\mu = \int_{\partial M} ds^\rho\, \epsilon_{\mu\rho} F^\mu$$



$$(d\xi_x, d\xi_y) = (ds^y, -ds^x)$$

ds is the tangent direction (反时针)

$$\Rightarrow \int_M d^2x\, \partial_\mu F^\mu = \int dx\, \epsilon_{yx} F^y + \int dy\, \epsilon_{xy} F^x = -\int dx\, F^y + \int dy\, F^x$$

$$\begin{cases} F^y = \frac{1}{2i}(F^z - F^{\bar z}) \\ F^x = \frac{1}{2}(F^z + F^{\bar z}) \end{cases}$$

逆变矢量假定其变换规律与坐标一样

$$\Rightarrow -\int \frac{1}{2}(dz + d\bar z)\frac{1}{2i}(F^z - F^{\bar z}) + \frac{1}{2i}(dz - d\bar z)\frac{1}{2}(F^z + F^{\bar z})$$

$$= \frac{1}{2i}\int dz\, F^{\bar z} - \frac{1}{2i}\int d\bar z\, F^z$$

$$\Rightarrow \boxed{\int_M d^2x\, \partial_\mu F^\mu = \frac{i}{2}\int_{\partial M} -dz\, F^{\bar z} + d\bar z\, F^z}$$

$$\boxed{\begin{aligned} &F^z = F^x + iF^y && F_z = \tfrac{1}{2}F^{\bar z} = \tfrac{1}{2}(F^x - iF^y) && F^x = F_z + F_{\bar z} \\ &F^{\bar z} = F^x - iF^y && F_{\bar z} = \tfrac{1}{2}F^z = \tfrac{1}{2}(F^x + iF^y) && F^y = i(F_z - F_{\bar z}) \end{aligned}}$$

Simaly for rank-two tensor. say E-M tensor

$$\boxed{\begin{aligned} &T^{zz} = T^{00} + i(T^{01} + T^{10}) - T^{11} && T_{zz} = \tfrac{1}{4}T^{\bar z\bar z} \\ &T^{\bar z\bar z} = T^{00} - i(T^{01} + T^{10}) - T^{11} && T_{\bar z\bar z} = \tfrac{1}{4}T^{zz} \\ &T^{z\bar z} = T^{00} - iT^{01} + iT^{10} + T^{11} && T_{z\bar z} = \tfrac{1}{4}T^{\bar z z} \\ &T^{\bar z z} = T^{00} + iT^{01} - iT^{10} + T^{11} && T_{\bar z z} = \tfrac{1}{4}T^{z\bar z} \end{aligned}}$$

We will use complex variable re organize the Ward identity

$$\partial^\mu \langle T_{\mu\nu}\, O \rangle = -\sum_i \delta(x-x_i)\, \partial_\nu \langle O \rangle \quad \Rightarrow$$

在用 τ, x 时. metric $\eta = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$. 不区分协, 逆变

$$\begin{cases} ① \quad \langle (\partial_0 T_{00} + \partial_1 T_{10}) O \rangle = -\sum \delta(x-x_i)\, \partial_{i,0} \langle O \rangle \\ ② \quad \langle (\partial_0 T_{01} + \partial_1 T_{11}) O \rangle = -\sum \delta(x-x_i)\, \partial_{i,1} \langle O \rangle \end{cases}$$

$$\begin{cases} T_{00} = T_{zz} + T_{\bar z\bar z} + T_{z\bar z} + T_{\bar z z} & \partial_0 = \partial_z + \partial_{\bar z} \\ T_{10} = i(T_{zz} - T_{\bar z\bar z} + T_{z\bar z} - T_{\bar z z}) & \partial_1 = i(\partial_z - \partial_{\bar z}) \\ T_{01} = i(T_{zz} - T_{\bar z\bar z} - T_{z\bar z} + T_{\bar z z}) & \\ T_{11} = -(T_{zz} + T_{\bar z\bar z} - (T_{z\bar z} + T_{\bar z z})) & \end{cases}$$

$$T_{00} - iT_{01} = 2(T_{zz} + T_{\bar z z}), \qquad T_{10} - iT_{11} = 2i(T_{zz} - T_{\bar z z})$$

$$① - i② \Rightarrow \langle \{\partial_0 (T_{00} - iT_{01}) + \partial_1 (T_{10} - iT_{11})\} O \rangle = -\sum \delta(x-x_i)(\partial_{i,0} - i\partial_{i,1}) \langle O \rangle$$

$$2\langle \{\partial_0 (T_{zz} + T_{\bar z z}) + i\partial_1 (T_{zz} - T_{\bar z z})\} O \rangle = -\sum_i \delta(x-x_i)\, 2\, \partial_{i z} \langle O \rangle$$

$$\langle \{(\partial_0 + i\partial_1) T_{zz} + (\partial_0 - i\partial_1) T_{\bar z z}\} O \rangle = -\sum_i \delta^{(2)}(x-x_i)\, \partial_{i,z} \langle O \rangle$$

$$\boxed{\begin{aligned} 2\pi \langle (\partial_{\bar z} T_{zz} + \partial_z T_{\bar z z}) O \rangle &= -\sum_i \partial_{\bar z} \frac{1}{z-\omega_i}\, \partial_{\omega_i} \langle O \rangle \\ 2\pi \langle (\partial_z T_{\bar z\bar z} + \partial_{\bar z} T_{z\bar z}) O \rangle &= -\sum_i \partial_z \frac{1}{\bar z-\bar\omega_i}\, \partial_{\bar\omega_i} \langle O \rangle \end{aligned}} \quad (*)$$

Similarly $\langle (T_{00} + T_{11}) O \rangle = -\sum_i \delta(x-x_i)\, \Delta_i \langle O \rangle$

$$2\langle (T_{z\bar z} + T_{\bar z z}) O \rangle = -\sum_i \delta(x-x_i)\, \Delta_i \langle O \rangle$$

$$\langle (T_{01} - T_{10}) O \rangle = -i \sum \delta(x-x_i)\, S^{10} \langle O \rangle$$

$$\langle -2i(T_{z\bar z} - T_{\bar z z}) O \rangle = -\sum_i \delta(x-x_i)\, S_i \langle O \rangle$$

$$2\pi \langle T_{\bar{z}z}\, O \rangle = -\sum_{i=1}^{n} \partial_{\bar{z}} \left(\frac{1}{z-\omega_i}\right) h_i \langle O \rangle$$

$$2\pi \langle T_{z\bar{z}}\, O \rangle = -\sum_{i=1}^{n} \partial_{z} \left(\frac{1}{\bar{z}-\bar{\omega}_i}\right) \bar{h}_i \langle O \rangle$$

$$2\pi\, \partial_z \langle T_{\bar{z}z}\, O \rangle = -\sum_{i=1}^{n} \partial_z \left[\partial_{\bar{z}} \left(\frac{1}{z-\omega_i}\right) h_i \langle O \rangle\right] = \sum_{i=1}^{n} \partial_{\bar{z}} \frac{1}{(z-\omega_i)^2} h_i \langle O \rangle$$

plug in (*) in page 4

$$2\pi \langle \partial_{\bar{z}} T_{zz}\, O(x_1 \cdots x_n) \rangle + \sum_{i=1}^{n} \partial_{\bar{z}} \frac{1}{z-\omega_i} \partial_{\omega_i} \langle O \rangle + \sum_{i=1}^{n} \partial_{\bar{z}} \frac{1}{(z-\omega_i)^2} h_i \langle O \rangle = 0$$

define $T(z\bar{z}) = -2\pi T_{zz}$, $\bar{T}(z\bar{z}) = -2\pi T_{\bar{z}\bar{z}}$

$$\begin{cases} \partial_{\bar{z}} \left\{ \langle T(z\bar{z})\, O(\omega_i \bar{\omega}_i) \rangle - \sum_{i=1}^{n} \left[\frac{1}{z-\omega_i} \partial_{\omega_i} \langle O(\omega_i \bar{\omega}_i) \rangle + \frac{h_i}{(z-\omega_i)^2} \langle O \rangle \right\} = 0 \\ \partial_{z} \left\{ \langle \bar{T}(z\bar{z})\, O(\omega_i \bar{\omega}_i) \rangle - \sum_{i=1}^{n} \left[\frac{1}{\bar{z}-\bar{\omega}_i} \partial_{\bar{\omega}_i} \langle O(\omega_i \bar{\omega}_i) \rangle + \frac{\bar{h}_i}{(\bar{z}-\bar{\omega}_i)^2} \langle O \rangle \right\} = 0 \end{cases}$$

These equations are holomorphic / anti-holomorphic relations, thus $T(z\bar{z})$ should be holomorphic as $T(z)$, and $\bar{T}(z\bar{z})$ should be $\bar{T}(\bar{z})$.

$$\Rightarrow \boxed{\begin{aligned} \langle T(z)\, O \rangle &= \sum_{i=1}^{n} \left\{ \frac{1}{z-\omega_i} \partial_{\omega_i} \langle O(\omega_i \bar{\omega}_i) \rangle + \frac{h_i}{(z-\omega_i)^2} \langle O \rangle \right\} + \text{regular} \\ \langle \bar{T}(\bar{z})\, O \rangle &= \sum_{i=1}^{n} \left\{ \frac{1}{\bar{z}-\bar{\omega}_i} \partial_{\bar{\omega}_i} \langle O(\omega_i \bar{\omega}_i) \rangle + \frac{\bar{h}_i}{(\bar{z}-\bar{\omega}_i)^2} \langle O \rangle \right\} + \text{regular} \end{aligned}}$$

Now let us combine translation, rotation, and dilation together

$$x'^{\mu} = x^{\mu} + \varepsilon^{\mu},$$ ε^{μ} also depends on x, thus this transformation is general. we only want ~~it~~ that is conformal.

the field transfs as $\phi'(x') = \phi(x) - \lambda(x)\Delta\phi - \frac{i}{2}\omega_{\rho\nu}(x) S^{\rho\nu}\phi.$

$\lambda(x)$ is the scale facter : $\lambda(x) = \frac{1}{2}\partial_\rho \varepsilon^\rho$ for 2D.

$$\omega_{\rho\nu}(x) = \varepsilon_{\rho\nu} \underbrace{\frac{1}{2}(\varepsilon^{\alpha\beta}\partial_\alpha \varepsilon_\beta)}_{\partial_0\varepsilon_1 - \partial_1\varepsilon_0}$$ ← 局部转角

$$\Rightarrow \phi'(x') = \phi(x') - \varepsilon^\mu \partial_\mu\phi - \left(\frac{1}{2}\partial_\rho\varepsilon^\rho\right)\Delta\phi - \frac{i}{2}\varepsilon_{\rho\nu}\left(\frac{1}{2}\varepsilon^{\alpha\beta}\partial_\alpha\varepsilon_\beta\right) S^{\rho\nu}\phi$$

ε-tensor 小位移

$$= \phi(x') \qquad + \delta\phi(x')$$

Now Let us consider the following expression

$$\partial_\mu[\varepsilon_\nu(x) T^{\mu\nu}(x)] = \varepsilon_\nu(x)\partial_\mu T^{\mu\nu} + \frac{1}{2}[\partial_\mu\varepsilon_\nu + \partial_\nu\varepsilon_\mu] T^{\mu\nu}$$

$$+ \frac{1}{2}(\partial_\mu\varepsilon_\nu - \partial_\nu\varepsilon_\mu) T^{\mu\nu}$$

Comment: For translation ε_ν itself is the small parameter, but 对放缩和旋转, $\partial_\mu\varepsilon_\nu \pm \partial_\nu\varepsilon_\mu$ 是相关的小量. 若 $\varepsilon_\nu(x)$ is a conformal transf $\Rightarrow$

$$\partial_\mu[\varepsilon_\nu T^{\mu\nu}(x)] = \varepsilon_\nu\partial_\mu T^{\mu\nu} + \frac{1}{2}(\partial_\rho\varepsilon^\rho)\eta_{\mu\nu} T^{\mu\nu} + \frac{1}{2}\varepsilon^{\alpha\beta}\partial_\alpha\varepsilon_\beta\,\varepsilon_{\mu\nu} T^{\mu\nu}$$

并不意味这是一个守恒流, 只是拆成三项而已!

then $\int_M dx^2 \, \partial_\mu \langle (T^{\mu\nu}(x)\, \varepsilon_\nu(x))\, O \rangle = \int_M dx^2 \, \partial_\mu (\langle T^{\mu\nu}(x))\, O \rangle\, \varepsilon_\nu(x)$

$$+ \int_M dx^2 \, \langle \eta_{\mu\nu} T^\mu{}_\nu \, O \rangle \left(\tfrac{1}{2}\partial_\rho \varepsilon^\rho\right) + \int_M dx^2 \, \langle \varepsilon_{\mu\nu} T^{\mu\nu} O \rangle \left(\tfrac{1}{2}\varepsilon^{\alpha\beta}\partial_\alpha \varepsilon_\beta\right)$$

M contains the points $x_1 \cdots x_n$, inside $O(x_1, \cdots x_n)$.

plug in the Ward identity derived before, we have

$$\int_M dx^2 \partial_\mu \langle (T^{\mu\nu}(x)\varepsilon_\nu(x))\, O \rangle = \int_M dx^2 \; - \sum_i \delta(x-x_i)\, \partial_{i,\nu} \langle O \rangle\, \varepsilon^\nu(x)$$

$$- \sum_i \delta(x-x_i)\, \Delta_i \langle O \rangle \left(\tfrac{1}{2}\partial_\rho \varepsilon^\rho\right) - i \sum_i \delta(x-x_i)\, S_i \langle O \rangle \left(\tfrac{1}{2}\varepsilon^{\alpha\beta}\partial_\alpha \varepsilon_\beta\right)$$

$$= - \sum_i \delta(x-x_i) \left[\varepsilon^\nu(x)\, \partial_{i,\nu} + \Delta_i \left(\tfrac{1}{2}\partial_\rho \varepsilon^\rho\right) + i\, S_i \left(\tfrac{1}{2}\varepsilon^{\alpha\beta}\partial_\alpha \varepsilon_\beta\right)\right] \langle O \rangle$$

$$= \langle \delta O \rangle_\varepsilon$$

where $\delta O(x_1, \cdots x_n) = \sum_{i=1}^{n} (\phi_1(x_1) \cdots \delta\phi_i(x_i) \cdots \phi(x_n)]$

and $\delta\phi_i(x_i) = \left\{ -\varepsilon^\mu \partial_{i,\nu} - \left(\tfrac{1}{2}\partial_\rho \varepsilon^\rho\right)\Delta_i - i\, S_i \left(\tfrac{1}{2}\varepsilon^{\alpha\beta}\partial_\alpha \varepsilon_\beta\right)\right\} \phi_i(x_i)$

$$\Rightarrow \quad \boxed{\int_M dx^2 \langle (\partial_\mu T^{\mu\nu}(x)\, \varepsilon_\nu(x))\, O \rangle = \langle \delta O \rangle_\varepsilon}$$

Next, we will change to complex variable

define $F^\mu = \langle T^{\mu\nu}(x)\, \varepsilon_\nu(x)\, O \rangle$

$$\int_M d^2x \; \partial_\mu \langle T^{\mu\nu}(x)\, \varepsilon_\nu(x)\, O \rangle = \frac{i}{2}\int_{\partial M} -dz \; F^{\bar z} + d\bar z \; F^{z}$$

$$F^z = F_x + iF_y = T^{xx}\epsilon_x + T^{xy}\varepsilon_y + i\left(T^{yx}\epsilon_x + T^{yy}\epsilon_y\right)$$

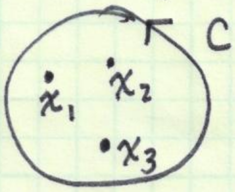
$$= (T_{zz} + T_{\bar z\bar z} + T_{z\bar z} + T_{\bar z z})(\epsilon_z + \epsilon_{\bar z}) + i(T_{zz} - T_{\bar z\bar z} - T_{z\bar z} + T_{\bar z z})(i)(\epsilon_z - \epsilon_{\bar z})$$

$$+ i(T_{zz} - T_{\bar z\bar z} + T_{z\bar z} - T_{\bar z z})(i)(\epsilon_z + \epsilon_{\bar z}) + i(-)(T_{zz} + T_{\bar z\bar z} - T_{z\bar z} - T_{\bar z z})(i)(\epsilon_z - \epsilon_{\bar z})$$

$$= 2[\epsilon_z + \epsilon_{\bar z}](T_{\bar z\bar z} + T_{\bar z z}) + 2(\epsilon_z - \epsilon_{\bar z})(T_{\bar z\bar z} - T_{\bar z z}) = 4\epsilon_z T_{zz} + 4\epsilon_{\bar z} T_{\bar z\bar z}$$

$$= T^{zz}\epsilon_z + T^{z\bar z}\epsilon_{\bar z}$$

$$\Rightarrow \int_M d^2x \;\; \partial_\mu \left\langle \left(T^{\mu\nu}(x)\, \varepsilon_\nu(x)\right) O \right\rangle = \frac{i}{2}\left[\int_{\partial M} d\bar z \left(T^{zz}\epsilon_z O + T^{z\bar z}\epsilon_{\bar z} O\right) - \int_{\partial M} dz \left(T^{\bar z\bar z}\epsilon_{\bar z} O + T^{\bar z z}\epsilon_z O\right)\right]$$



(*) comment

$T^{z\bar z}(z)\, O(\omega_i \bar\omega_i)$ 和 $T^{zz} \cdot O$ 的行为完全不同.

$$2\pi \langle T_{z\bar z}\, O \rangle = -\sum_{i=1}^{n} \partial_{\bar z} \frac{1}{z_i - \omega_i} h_i \langle O \rangle \sim \sum_{i=1}^{N} \delta^{(2)}(x - \omega_i)$$

it's only nonzero at points ω_i. ∂M doesn't pass these points $\Rightarrow$ $T_{z\bar z} O$ and $T^{\bar z z} O$ do not contribute to the $\oint_{\partial M}$. But $T_{z\bar z}$'s relation involves $\partial_z T_{zz}$, and $T_{zz} O$ itself has the part of $\frac{1}{z-\omega_i}$, thus T^{zz} and $T^{\bar z\bar z}$ do contribute $\Rightarrow$

$$\boxed{\begin{aligned} \delta_{\epsilon\bar\epsilon}\langle O \rangle &= \frac{i}{2}\oint_{\partial M} -dz \left\langle T^{\bar z\bar z} O \right\rangle \epsilon_{\bar z} + \frac{i}{2}\oint_{\partial M} d\bar z \left\langle T^{zz} O \right\rangle \epsilon_z \\ \text{or} \quad &= -\frac{1}{2\pi i}\oint_{\partial M} dz\, \epsilon(z) \langle T(z)\, O \rangle + \frac{1}{2\pi i}\oint_{\partial M} d\bar z\, \bar\varepsilon(\bar z) \langle \bar T(\bar z)\, O \rangle \end{aligned}}$$

注: $T^{\bar z\bar z}\epsilon_{\bar z} = 2\, T_{zz}\, \epsilon^{z} = -\frac{1}{\pi} T(z)\, \epsilon(z)$.

If O is primary. $\langle T(z)\, O\rangle \sim \sum_{i=1}^{n} \frac{1}{z_i-\omega_i} \partial_{\omega_i}\langle O\rangle + \frac{h_i}{(z_i-\omega_i)^2}\langle O\rangle$

$$\boxed{\delta_\epsilon \langle O\rangle = -\frac{1}{2\pi i}\oint_c dz\, \epsilon(z) \langle T(z)\, O\rangle = -\sum_i \Big[\underbrace{\epsilon(\omega_i)\,\partial_{\omega_i}}_{\text{translation}} + \underbrace{\partial_{\omega_i}\varepsilon(\omega_i)\, h_i}_{\text{strain tensor}}\Big]\langle O\rangle}$$

对比 P8 (第一部分) $\delta_{\epsilon\epsilon}\phi = -(h\partial_z\varepsilon + \varepsilon\partial_z + \bar{h}\partial_{\bar{z}}\bar{\varepsilon} + \bar{\varepsilon}\partial_{\bar{z}})\phi$

They are ~~consistent~~!

For infinitesimal conformal ~~transformation~~ of $f(z) = \frac{(1+\alpha)z+\beta}{\gamma z + 1-\alpha}$

β: ~~translation~~, γ: special conf

α: dilation

This is a true symmetry, thus

$\delta_\epsilon \langle O\rangle = 0$. The above boxed formula $\delta_\epsilon\langle O\rangle$ apply from arbitrary small ~~trans~~formation. But for that associated with $f(z)$, i.e. $\varepsilon(z) = \beta + 2\alpha z - \gamma z^2$, we should have $\delta_\epsilon\langle O\rangle = 0$.

for conformal field theory. $\Rightarrow$

$$\boxed{\begin{aligned}
&\sum \partial_{\omega_i} \langle \phi_1(\omega_1)\cdots\phi_n(\omega_n)\rangle = 0 && \text{for } \epsilon = \text{const} \\
&\sum [\omega_i\partial_{\omega_i} + h_i] \langle \phi_1(\omega_1)\cdots\phi_n(\omega_n)\rangle = 0 && \text{for } \epsilon = 2\alpha z \\
&\sum (\omega_i^2\partial_{\omega_i} + 2\omega_i h_i) \langle \phi_1(\omega_1)\cdots\phi_n(\omega_n)\rangle = 0 && \text{for } \epsilon = z^2
\end{aligned}}$$

~~consistently~~ with previous results!

we have not explicitly introduce OPE yet. The above result

$$\langle T(z)\, O \rangle = \sum_{i=1}^{n} \frac{1}{z-\omega_i} \partial_{\omega_i} \langle O \rangle + \frac{h_i}{(z-\omega_i)^2} \langle O \rangle + \text{reg.}$$

$$\langle \bar{T}(\bar{z})\, O \rangle = \sum_{i=1}^{n} \frac{1}{\bar{z}-\bar{\omega}_i} \partial_{\bar{\omega}_i} \langle O \rangle + \frac{\bar{h}_i}{(\bar{z}-\bar{\omega}_i)^2} \langle O \rangle + \text{reg.}$$

This can be denoted as OPE

$$\boxed{\begin{aligned} T(z)\,\phi(\omega,\bar{\omega}) &\sim \frac{h}{(z-\omega)^2}\,\phi(\omega,\bar{\omega}) + \frac{1}{z-\omega}\,\partial_\omega \phi(\omega,\bar{\omega}) \\ T(\bar{z})\,\phi(\omega,\bar{\omega}) &\sim \frac{\bar{h}}{(\bar{z}-\bar{\omega})^2}\,\phi(\omega,\bar{\omega}) + \frac{1}{\bar{z}-\bar{\omega}}\,\partial_{\bar{\omega}} \phi(\omega,\bar{\omega}) \end{aligned}}$$

"$\sim$" means equality up to regular term as $z \to \omega$. In general we can write

$$A(z)\, B(\omega) = \sum_{n=-\infty}^{N} \frac{1}{(z-\omega)^n} \{A B\}_n(\omega) .$$

Cetain operaters regular at ω.

Rigorously speaking, the quantities here are not operaters, but field in correlation functions.

Central charge:

$T(z)$ is not quite a primary field,

$$T(z)\,T(w) \sim \frac{c/2}{(z-w)^4} + \frac{2\,T(w)}{(z-w)^2} + \frac{1}{z-w}\,\partial_w T(w)$$

⊛ 从 leading term's power, we can infer the dimension of $T(z)$ is 2. This can also be seen from $T(z)\,\phi(w,\bar{w}) \sim \frac{h}{(z-w)^2}\,\phi(w,\bar{w})$. But $T(z)T(w)$ has an extra leading term $\frac{c/2}{(z-w)^4}$.

$$\delta_\epsilon T(w) = -\frac{1}{2\pi i}\oint dz\,\epsilon(z)\,T(z)\,T(w) = -\frac{1}{2\pi i}\oint dz\,\epsilon(z)\left\{\frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{1}{z-w}\,\partial_w T(w)\right\}$$

$$\boxed{\delta_\epsilon T(w) = \boxed{-\frac{c}{12}\,\partial_w^3\,\epsilon(w)} - 2T(w)\,\partial_w \epsilon(w) - \epsilon(w)\,\partial_w T(w)}$$

对比 Primary field | $\delta_{\epsilon\bar{\epsilon}}\,\phi = -(h\,\partial_z \varepsilon + \varepsilon\,\partial_z)\,\phi(z,\bar{z}) - (\bar{h}\,\partial_{\bar{z}}\bar{\varepsilon} + \bar{\varepsilon}\,\partial_{\bar{z}})\,\phi(z,\bar{z})$

extra term.

If for finite conformal transformation $z \to w(z)$, $T(z)$ transforms as

$$\boxed{T'(w) = \left(\frac{dw}{dz}\right)^{-2}\left[T(z) - \frac{c}{12}\{w;z\}\right]}$$

$$\boxed{\text{where } \{w;z\} = \frac{d^3w/dz^3}{dw/dz} - \frac{3}{2}\left(\frac{d^2w/dz^2}{dw/dz}\right)^2}$$ ← Schwarzian derivative

⊗ comment: The extra "c"-term show the difference betwee T and primary field.

For a primary field $\phi'(w(z),\bar{w}(\bar{z})) = \left(\frac{dw}{dz}\right)^{-h}\left(\frac{d\bar{w}}{d\bar{z}}\right)^{-\bar{h}}\phi(z,\bar{z})$.

⊛ let us check infinitsimal mapping $\omega = z + \epsilon(z)$

$$\{z+\epsilon(z); z\} = \frac{\partial_z^3 \epsilon}{1+\partial_z\epsilon} - \frac{3}{2}\left(\frac{\partial_z^2\epsilon}{1+\partial_z\epsilon}\right)^2 \simeq \partial_z^3\epsilon \text{ to 1st order in } \epsilon.$$

thus $T'(\omega) = T'(z+\epsilon(z)) = (1-2\partial_z\epsilon)\left(T(z) - \frac{1}{12}c\,\partial_z^3\epsilon\right)$

$$= T'(z) + \epsilon(z)\,\partial_z T$$

可以忽略 the difference between T' and T

$$\Rightarrow T'(z) + \epsilon(z)\partial_z T = T(z) - 2\partial_z\epsilon\, T(z) - \frac{1}{12}c\,\partial_z^3\epsilon$$

$$\Rightarrow \delta_\epsilon T(z) = T'(z) - T(z) = -\epsilon(z)\partial_z T - 2\partial_z\epsilon\, T(z) - \frac{c}{12}\partial_z^3\epsilon. \checkmark$$

⊛ For a global conformal mapping $\omega(z) = \frac{az+b}{cz+d}$ with $ad-bc=1$. the Schwarzian derivative $=0$, thus, $T'(\omega) = \left(\frac{d\omega}{dz}\right)^{-2} T(z)$ for global conformal mapping / but not for local conformal mapping!

Check: $\left.\begin{aligned}\omega &= z + c\\ \omega &= \lambda z\end{aligned}\right\} \Rightarrow d^2\omega/dz^2 = d^3\omega/dz^3 = 0 \checkmark$

$$\frac{1}{\omega} = \frac{1}{z} + C \Rightarrow \frac{d\omega}{dz} = \frac{\omega^2}{z^2}\frac{d(1/\omega)}{d(1/z)} = \frac{\omega^2}{z^2}$$

$$\frac{d^2\omega}{dz^2} = -\frac{1}{z^2}\frac{d}{d(\frac{1}{z})}\left[\frac{(1/z)^2}{(1/\omega)^2}\right] = -\frac{1}{z^2}\left[\frac{2(\frac{1}{z})}{(1/\omega)^2} + \frac{-2(\frac{1}{z})^2}{(1/\omega)^3}\right]$$

$$= -\frac{2\left(\frac{1}{z}\right)^3 \cdot C}{\left(\frac{1}{\omega}\right)^3} = -2C\left(\frac{\omega}{z}\right)^3$$

$$\frac{d^3\omega}{dz^3} = \frac{2C}{z^2}\cdot\frac{d}{d(\frac{1}{z})}\left[\frac{(\frac{1}{z})^3}{(1/\omega)^3}\right]$$

$$= \frac{2C}{z^2}\left[\frac{3(\frac{1}{z})^2(\frac{1}{\omega}-\frac{1}{z})}{(1/\omega)^4}\right] = 6C^2\left(\frac{\omega}{z}\right)^4$$

$$\Leftarrow \quad \{\omega, z\} = \frac{6C^2\left(\frac{\omega}{z}\right)^4}{\left(\omega/z\right)^2} - \frac{3}{2}\,4C^2\left(\frac{\omega}{c}\right)^2 = 0$$

对平移, 旋转, 放缩, 以及 special conf, $\{\omega, z\} = 0$.

For functions $u = u(\omega) = u(\omega(z))$, we have

$$\boxed{\{u;z\} = \{\omega;z\} + \left(\frac{d\omega}{dz}\right)^2\{u;\omega\}}$$

if $\{\omega,z\}=0$, and $\{u,\omega\}=0$, $\Rightarrow \{u;z\}=0$, thus all the global conformal ~~transf~~ gives zero Schwartzian derivative.

Proof:

$$\{u;z\} = \frac{d^3u/dz^3}{du/dz} - \frac{3}{2}\left(\frac{d^2u/dz^2}{du/dz}\right)^2$$

$$\frac{du}{dz} = \frac{du}{d\omega}\frac{d\omega}{dz}, \quad \frac{d^2u}{dz^2} = \frac{d^2u}{d\omega^2}\left(\frac{d\omega}{dz}\right)^2 + \frac{du}{d\omega}\frac{d^2\omega}{dz^2}$$

$$\left[\frac{d^2u/dz^2}{du/dz}\right]^2 = \left[\frac{d^2u/d\omega^2}{du/d\omega}\frac{d\omega}{dz} + \frac{d^2\omega/dz^2}{d\omega/dz}\right]^2 = \left(\frac{d\omega}{dz}\right)^2\left(\frac{d^2u/d\omega^2}{du/d\omega}\right)^2 + \overbrace{\left(\frac{d^2\omega/dz^2}{d\omega/dz}\right)^2 + 2\,\frac{\frac{d^2u}{d\omega^2}\frac{d^2\omega}{dz^2}}{du/d\omega}}$$

$$\frac{d^3u}{dz^3} = \frac{d}{dz}\left(\frac{d^2u}{d\omega^2}\left(\frac{d\omega}{dz}\right)^2\right) + \frac{d}{dz}\left(\frac{du}{d\omega}\frac{d^2\omega}{dz^2}\right)$$

$$= \frac{d^3u}{d\omega^3}\left(\frac{d\omega}{dz}\right)^3 + \underbrace{2\frac{d^2u}{d\omega^2}\frac{d^2\omega}{dz^2}\frac{d\omega}{dz} + \frac{d^2u}{d\omega^2}\frac{d^2\omega}{dz^2}\frac{d\omega}{dz}}_{\text{同类项}} + \frac{du}{d\omega}\frac{d^3\omega}{dz^3}$$

相消

$$\Rightarrow \frac{d^3u/dz^3}{du/dz} = \underbrace{\frac{\frac{d^3u}{d\omega^3}}{du/d\omega}\left(\frac{d\omega}{dz}\right)^2 + \frac{d^3\omega/dz^3}{d\omega/dz}} + 3\,\frac{\frac{d^2u}{d\omega^2}\cdot\frac{d^2\omega}{dz^2}}{du/d\omega}$$

加在一起，并注意 $-3/2$ 的系数，即证 ✓.

(*) Consider the ~~transf~~ $z \to \omega \to u$, we should have $u(\omega(z))$

$$\boxed{T''(u) = \left(\frac{du}{dz}\right)^{-2}\left[T(z) - \frac{c}{12}\{u;z\}\right]}$$ ← 即群的结合律

Proof: $T''(u) = \left(\frac{du}{d\omega}\right)^{-2}\left[T'(\omega) - \frac{c}{12}\{u,\omega\}\right] = \left(\frac{du}{d\omega}\right)^{-2}\left[\left(\frac{d\omega}{dz}\right)^{-2}\left(T(z) - \frac{c}{12}\{\omega;z\}\right) - \frac{c}{12}\{u,\omega\}\right]$

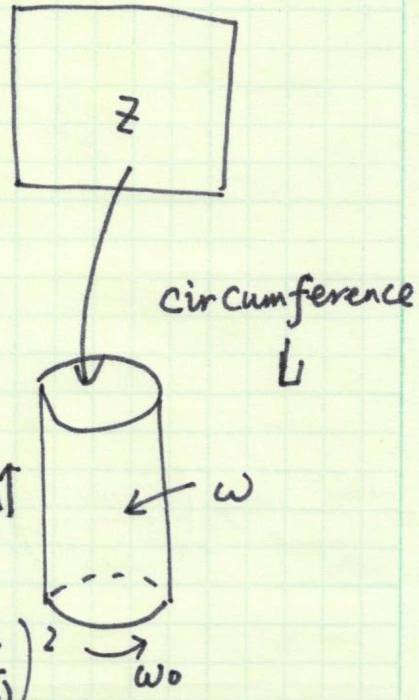
$$= \left(\frac{du}{dz}\right)^{-2}\left[T(z) - \frac{c}{12}\left\{\{\omega;z\} + \left(\frac{d\omega}{dz}\right)^2\{u;\omega\}\right\}\right] = \left(\frac{du}{dz}\right)^{-2}\left[T(z) - \frac{c}{12}\{u;z\}\right] \checkmark$$

another useful formula. for $z \to \omega \to z$, we have $u = z \Rightarrow \{u, z\} = 0$

$$\Rightarrow \{\omega, z\} = -\left(\frac{d\omega}{dz}\right)^2 \{z, \omega\} \Rightarrow$$

$$\boxed{T'(\omega) = \left(\frac{d\omega}{dz}\right)^{-2} T(z) + \frac{c}{12}\{z; \omega\}}$$

* Physical meaning of conformal charge "C".



plane -> cylinder

consider the mapping $z \to \omega = \frac{L}{2\pi} \ln z$

$\mathrm{Im} \ln z \in [0, 2\pi)$, $\Rightarrow \mathrm{Im}\, \omega \in [0, L)$.

$$\frac{d\omega}{dz} = \frac{L}{2\pi z} \Rightarrow \{\omega; z\} = \frac{\frac{L}{\pi}\left(\frac{1}{z}\right)^3}{\frac{L}{2\pi}\left(\frac{1}{z}\right)} - \frac{3}{2}\left(\frac{\frac{L}{2\pi}\frac{1}{z^2}}{\frac{L}{2\pi}\left(\frac{1}{z}\right)}\right)^2$$

$$\frac{d^2\omega}{dz^2} = \frac{-L}{2\pi}\frac{1}{z^2}, \quad \frac{d^3\omega}{dz^3} = \frac{L}{\pi}\frac{1}{z^3}$$

$$= 2\left(\frac{1}{z}\right)^2 - \frac{3}{2}\left(\frac{1}{z}\right)^2 = \frac{1}{2z^2}$$

$$\Rightarrow \boxed{T_{cyl}(\omega) = \left(\frac{2\pi}{L}\right)^2 \left[T_{plane}(z)\, z^2 - \frac{c}{24} \right] \quad \text{under } \omega = \frac{L}{2\pi}\ln z}$$

if we assume $\langle T_{plane} \rangle = 0$, we get an nonzero vacuum energy

for a cylinder $\langle T_{cyl}(\omega) \rangle = -\frac{c\pi^2}{6L^2}$ ← it's negative

finite size -> related to Casimir energy.

Using the result of

$$\delta F = -\frac{1}{2}\int d^2x \sqrt{g}\ \delta g_{\mu\nu} \langle T^{\mu\nu}\rangle,$$ and consider a cylinder

Let us do an scaling of the circumferenc $L \to L + dL = (1+\epsilon)L$.

since $\omega = \omega^0 + i\omega^1$, ~~or~~ and this ~~transf~~ corresponds to $\begin{cases} \omega^0 \to (1+\epsilon)\omega^0 \\ \omega^1 \to \omega^1 \end{cases}$

The infinitesimal elements $\epsilon^\mu = \delta_{\mu,0}\, \omega^0 \epsilon \Rightarrow \frac{\partial \varepsilon^0}{\partial \omega^0} = \epsilon$

$$\Rightarrow \delta g_{\mu\nu} = -2\epsilon\, \delta_{\mu 0}\delta_{\nu 0} = -\left[\partial_\mu \varepsilon_\nu + \partial_\mu \varepsilon_0\right]$$

to maintain ds^2 unchanged!

其实 $ds^2 = \underbrace{[1-\epsilon]^2}_{\delta g_{00}}\left\{(1+\epsilon)^2 (d\omega^0)^2\right\} + (d\omega^1)^2$

$$\Rightarrow \delta F = \int d\omega^0 d\omega^1 \underbrace{(+)}_{\sqrt{g}\approx 1+O(\varepsilon)} \varepsilon \langle T^{00}\rangle$$

$$\langle T^{00}\rangle = \langle T_{zz}\rangle + \langle T_{\bar z\bar z}\rangle = -\frac{1}{\pi}\langle T(z)\rangle = \frac{\pi C}{6L^2}$$

定义 $2T_{zz} = -\frac{1}{\pi}T(z)$

$$= \int d\omega^0 d\omega^1 \frac{\pi C}{6L^2}\frac{\delta L}{L}$$

$$= L_{\omega_1}\cdot \frac{\pi C}{6L^2}\delta L$$

单位长度 $\delta\left(\frac{F}{L_{\omega_1}}\right) = \frac{\pi C}{6L^2}\delta L$, 如果在 $L\to\infty$, 还有 f_0 的真空能 $\langle T^{00}\rangle$

this result is modified $\delta\left(\frac{F}{L_{\omega_1}}\right) = \int d\omega^0 \left(f_0 + \frac{\pi C}{6L^2}\right)\frac{\delta L}{L}$

$$= \left(f_0 + \frac{\pi C}{6L^2}\right)\delta L$$

宇宙常数

$$\Rightarrow \boxed{\frac{F}{L_{\omega_1}} = f_0 L - \frac{\pi C}{6L} + \text{const}}$$