Noether theorem:

Consider a field theory $\mathcal{L}[\phi_i(x), \partial_\mu \phi_i(x)]$ with the action

$$S[\phi] = \int d^dx \; \mathcal{L}[\phi_i(x), \partial_\mu \phi_i(x)] \,.$$

Let us apply a transformation $\begin{cases} x'^\mu = x^\mu + \delta x^\mu \\ \phi_i'(x') = \phi_i(x) + \delta\phi_i(x) \end{cases}$.

we use $\{\omega_a\}$ as a set of small parameters for the above transf

$$\delta x^\mu = \omega_a \frac{\delta x^\mu}{\delta \omega_a} \,, \qquad \delta\phi_i = \omega_a \frac{\delta \phi_i(x)}{\delta \omega_a} \,.$$

under this transf

$$S[\phi'] = \int d^dx \, \mathcal{L}[\phi'(x), \partial_\mu \phi'(x)] = \int d^dx' \, \mathcal{L}[\phi'(x'), \partial'_\mu \phi'(x')]$$

⊛ just change dummy variable $x \to x'$

$$\frac{\partial' x^\mu}{\partial x^\nu} = 1 + \partial_\mu(\delta x^\nu) \Rightarrow \left|\frac{\partial' x^\mu}{\partial x^\nu}\right| = 1 + \partial_\mu(\delta x^\mu)$$

$$\partial'_\mu \phi'(x') = \frac{\partial x^\nu}{\partial x'^\mu} \partial_\nu \phi'(x') = [\delta^\nu{}_\mu - \partial_\mu \delta x^\nu][\partial_\nu \phi + \partial_\nu \delta\phi(x)]$$

$$= \partial_\mu \phi + \partial_\mu \delta\phi - \partial_\mu(\delta x^\nu)\, \partial_\nu \phi$$

$$\Rightarrow S[\phi'] = \int d^dx \left|\frac{\partial' x^\mu}{\partial x^\nu}\right| \mathcal{L}[\phi + \delta\phi(x), \partial_\mu\phi + \partial_\mu \delta\phi - \partial_\mu \delta x^\nu \partial_\nu \phi]$$

$$= S[\phi] + \int d^dx \; \frac{\partial \mathcal{L}}{\partial \phi} \delta\phi(x) + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\mu \delta\phi + \partial_\mu(\delta x^\mu)\mathcal{L} - \frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)} \partial_\mu(\delta x^\nu) \partial_\nu \phi$$

$$\Rightarrow \delta S = \int d^dx \; \frac{\partial \mathcal{L}}{\partial \phi_i} \frac{\delta \phi_i(x)}{\delta \omega_a} \omega_a + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\mu \left(\omega_a \frac{\delta \phi}{\delta \omega_a}\right)$$

$$+ \partial_\mu\left(\omega_a \frac{\delta x^\mu}{\delta \omega_a}\right) \mathcal{L} - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\nu \phi \; \partial_\mu \left[\omega_a \frac{\delta x^\nu}{\delta \omega_a}\right]$$

$$= \int d^dx \;\; \omega_a \left[\frac{\partial \mathcal{L}}{\partial \phi_i} \frac{\delta \phi_i}{\delta \omega_a} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \partial_\mu \left(\frac{\delta \phi_i}{\delta \omega_a}\right) + \partial_\mu \left(\frac{\delta x^\mu}{\delta \omega_a}\right) \mathcal{L} \right.$$

$$\left. - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\nu \phi \; \partial_\mu \left(\frac{\delta x^\nu}{\delta \omega_a}\right)\right]$$

$$- \left[\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\nu \phi - \delta^\mu_\nu \mathcal{L} \right) \frac{\delta x^\nu}{\delta \omega_a} - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \frac{\delta \phi}{\delta \omega_a} \right] \partial_\mu \omega_a$$

comment ① we do not assume ϕ is the saddle point solution, thus we cannot use Euler- Langrang equation!

② We do assume, $S[\phi]$ has the symmetry under a rigid ~~transf~~, i.e. ω_a is a constant. In this case, the linear term of ω_a should vanish.

$$\Rightarrow \boxed{\delta S = - \int d^dx \;\; j^\mu_a \, \partial_\mu \omega_a(x) \xrightarrow[\text{up to partial derivative}]{} \int d^dx \; \partial_\mu j^\mu_a(x) \, \omega_a(x)}$$

with $j_{a,\mu}(x) = \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\nu \phi - \delta^\mu_\nu \mathcal{L} \right] \frac{\delta x^\nu}{\delta \omega_a} - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \frac{\delta \phi}{\delta \omega_a}$

③ although by partial derivative, we have $\delta S = \int d^dx \; \partial_\mu j(x) \omega_a(x)$, we can not get $\partial_\mu j_\mu = 0$, because we cannot say for arbitany $w(x)$, we have $\delta S = 0$. It just gives $\int d^dx\, \partial_\mu j^\mu = 0$, not we want!

Then why we can say that the first term linear to ω_a in page 2 vanishes? It's really a local property of symmetry.

$$(\Delta X)^d \mathcal{L} \longrightarrow (\Delta X')^d \mathcal{L}[\phi'(x'), \partial'_\mu \phi(x)] \text{ should be the same.}$$

This is a property from point $x \to x'$, $\phi(x) \to \phi'(x')$, not a result after integrating. For example, we check

a) a complex scalar field, $\mathcal{L} = (\partial_\mu \phi^*)(\partial^\mu \phi) + m\phi^*\phi$. The interal $U(1)$ sym

$$\begin{cases} x'^\mu = x^\mu \\ \phi'(x') = \phi(x) + i\alpha\phi(x) \\ \phi'^*(x') = \phi^*(x) - i\alpha\phi^*(x) \end{cases} \Rightarrow \frac{\delta x^\mu}{\delta \alpha} = 0, \quad \frac{\delta \phi}{\delta \alpha} = i\phi, \quad \frac{\delta \phi^*}{\delta \alpha} = -i\phi$$

$$\Rightarrow \frac{\partial \mathcal{L}}{\partial \phi}\frac{\delta \phi}{\delta \alpha} + \frac{\partial \mathcal{L}}{\partial \phi^*}\frac{\delta \phi^*}{\delta \alpha} = im\phi^*\phi - im\phi^*\phi = 0$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\partial_\mu\left(\frac{\delta \phi}{\delta \alpha}\right) + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^*)}\partial\left(\frac{\delta \phi^*}{\delta \alpha}\right) = (\partial_\mu \phi)^* i \partial_\mu \phi + \partial_\mu \phi (-i\partial_\mu \phi)^* = 0$$

b) Check rotation for scalar field theory:

$$\begin{cases} x'^\mu = x^\mu + \omega^\mu{}_\nu x^\nu \\ \dfrac{\delta x^\mu}{\delta \omega^\lambda{}_\delta} = (\delta^\mu{}_\lambda x^\sigma - \delta^\mu{}_\delta x^\lambda) \\ \delta\phi / \delta\omega^\lambda{}_\delta = 0 \end{cases}$$

$$\partial_\mu\left(\frac{\delta x^\mu}{\delta \omega^\lambda{}_\delta}\right) = [\delta^\mu_\lambda \delta^\delta_\mu - \delta^\mu_\delta \delta^\lambda_\mu] = 0$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\partial_\nu \phi\, \partial_\mu\left[\frac{\delta x^\nu}{\delta \omega^\lambda{}_\delta}\right] = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\partial_\nu \phi\, \partial_\mu[\delta^\mu_\lambda x^\delta - \delta^\nu_\delta x^\lambda]$$

$$= \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\partial_\nu \phi\, (\delta^\mu_\lambda \delta^\delta_\mu - \delta^\nu_\delta \delta^\lambda_\mu) = \left[\frac{\partial \mathcal{L}}{\partial(\partial_\delta \phi)}\partial_\lambda \phi - \frac{\partial \mathcal{L}}{\partial(\partial_\lambda \phi)}\partial_\delta \phi\right]$$

$$= 0$$

So far, we only consider $S[\phi]$, not $Z = \int D\phi \, e^{-S[\phi]}$, i.e, we have not average over all the field configuration yet. We will see how the conservation law appears after averaging over $[d\phi]$. We are not just consider the saddle point field configuration, but the average value, $\longrightarrow$ Ward identities!

Now let us check under the transformation on page 1.

$$\phi'(x^\mu) = \phi(x^\mu - \frac{\delta x^\mu}{\delta \omega^a}\omega^a) + \omega_a \frac{\delta\phi(x)}{\delta\omega}$$

$$= \phi(x^\mu) - \omega^a \frac{\delta x^\mu}{\delta\omega^a}\partial_\mu\phi + \omega_a\frac{\delta\phi}{\delta\omega} = \phi(x^\mu) \underbrace{- i\omega_a G_a \phi(x)}$$

$$\Rightarrow \boxed{-i \, G_a \phi(x) = -\frac{\delta x^\mu}{\delta\omega^a}\partial_\mu\phi + \frac{\delta\phi}{\delta\omega^a}}$$

Consider a set of variables $O = \phi_1(x_1) \cdots \phi_n(x_n)$

under the above ~~transf~~ $O(x_1 \cdots x_n) \rightarrow O + \delta O$

$$\boxed{\delta O = -i\sum_{i=1}^{n} [\phi_1(x_1) \cdot G_a \phi_i(x_i) \cdots \phi_n(x_n)] \, \omega_a(x_i)}$$

$$\langle O \rangle = \frac{1}{Z}\int [d\phi] \, O[\phi] \, e^{-S[\phi]} = \frac{1}{Z}\int [d\phi'] \, O[\phi'] \, e^{-S[\phi']}$$

change dumy variable

$$\langle O(x_1 \cdots x_n)\rangle = \frac{1}{Z}\int [D\phi] \, O[\phi_1'(x_1) \cdots \phi_n'(x_n)] \, e^{-S[\phi']}$$

assume integral measure the same as $[d\phi']$

$x_1, \cdots x_n$ here are external parameters, not dummy variables, which do not change.

$$S[\phi'] = \int dx' \, \mathcal{L}[\phi'(x'), \partial'_x \phi'(x')] = \int dx \, \mathcal{L}[\phi'(x), \partial_x \phi'(x)]$$

$$= S[\phi] + \int dx \, \partial_\mu j^\mu_a \, \omega_a(x)$$

$$\Rightarrow \langle O \rangle = \frac{1}{Z} \int [d\phi] \, (O + \delta O) \, e^{-S[\phi]} \left(1 - \int dx \, \partial_\mu j^\mu_a \, \omega_a(x)\right)$$

$$\Rightarrow \boxed{\langle \delta O \rangle = \int dx \, \partial_\mu \langle j^\mu_a(x) \, O \rangle \, \omega_a(x)}$$

plug in $\delta O = -i \int dx \sum_{i=1}^{n} \{\phi_1(x_1) \cdots G_a \phi_i(x_i) \cdots \phi_n(x_n)\} \, \delta(x - x_i) \, \omega_a(x)$

equivalent to the boxed expression in Page 4

$$\Rightarrow \boxed{\partial_\mu \langle j^\mu_a(x) \, O \rangle = -i \sum_{i=1}^{n} \delta(x - x_i) \langle \phi_1(x_1) \cdots G_a \phi_i(x_i) \cdots \phi_n(x_n) \rangle}$$

Now we will extensively explore the consequence of this expression for translation, Lorentz, dilation, spec. conformal transf's.

① Translation $\longrightarrow$ energy-momentum tensor

$$x'^{\mu} = x^{\mu} + \epsilon^{\mu} \quad \Rightarrow \quad \boxed{\frac{\delta x'^{\mu}}{\delta \varepsilon^{\nu}} = \delta^{\mu}_{\ \nu}}$$

$$\phi'(x'^{\mu}) = \phi(x) = \phi(x'^{\mu} - \varepsilon^{\mu}) = \phi(x'^{\mu}) - \varepsilon^{\mu}\partial_{\mu}\phi$$

$$\Rightarrow \boxed{\frac{\delta \phi}{\delta \varepsilon^{\mu}} = 0, \quad -iG_{\nu}\phi = -\frac{\delta x^{\mu}}{\delta \varepsilon^{\nu}}\partial_{\mu}\phi = -\partial_{\nu}\phi}$$

$$T_{c\ \ \nu}^{\ \mu} = \left[\frac{\partial \mathcal{L}}{\partial(\partial_{\mu}\phi)}\partial_{\lambda}\phi - \delta^{\mu}_{\lambda}\mathcal{L}\right]\delta^{\lambda}_{\ \nu} \quad \frac{\partial \mathcal{L}}{\partial(\partial_{\mu}\phi)}\partial_{\nu}\phi - \delta^{\mu}_{\nu}\mathcal{L}$$

we call this cannonical EM tensor, and later it may be augment by requiring certain properties, such as symmetric, tranceless, etc.

$$\Rightarrow \frac{\partial}{\partial x^{\mu}}\langle T^{\mu}_{\ \nu}(x)\,\phi_1(x_1)\cdots\phi_n(x_n)\rangle = -i\sum_{i=1}^{n}\delta(x-x_i)\,\partial_{i,\nu}\langle\phi_1(x_1)\cdots\phi_n(x_n)\rangle$$

$$\Rightarrow \boxed{\begin{aligned} T_c^{\mu\nu} &= -g^{\mu\nu}\mathcal{L} + \frac{\partial \mathcal{L}}{\partial(\partial_{\mu}\phi)}\partial^{\nu}\phi \\ \partial_{\mu}\langle T^{\mu\nu}(x)\, O\rangle &= -\sum_i \delta(x-x_i)\,\partial_i^{\nu}\langle O\rangle \end{aligned}}$$

§ Another way to define E-M tensor [Big-yellow-book (BYB P49)]:

Acording to $\delta S = \int d^d x\ j^{\mu}_{\ a}\ \partial_{\mu}\omega_a(x)$. Now $\omega_{\nu}(x) = \varepsilon_{\nu}(x)$

$$\Rightarrow \delta S = \int d^d x\ T^{\mu\nu}\,\partial_{\mu}\varepsilon_{\nu}(x) = \frac{1}{2}\int d^d x\ T^{\mu\nu}(\partial_{\mu}\varepsilon_{\nu} + \partial_{\nu}\varepsilon_{\mu})$$

(assuming $T^{\mu\nu}$ is symmetric)

check metric $g'_{\mu\nu} = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta} = (\delta^\alpha_\mu - \partial_\mu \varepsilon^\alpha)(\delta^\beta_\nu - \partial_\nu \varepsilon^\beta) g_{\alpha\beta}$

$$= g_{\mu\nu} - (\partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu)$$

$$\Rightarrow \boxed{\delta S = -\frac{1}{2}\int d^d x \; T^{\mu\nu}\, \delta g_{\mu\nu}}$$

§2. Lorentz transformation $x'^\mu = x^\mu + \omega^\mu{}_\nu x^\nu$

$$= x^\mu + g^{\mu\rho}\omega_{\rho\nu} x^\nu$$

with $\omega_{\rho\nu} = -\omega_{\nu\rho}$

$$\Rightarrow \underline{\frac{\delta x^\mu}{\delta \omega_{\rho\nu}} = g^{\mu\rho} x^\nu - g^{\mu\nu} x^\rho}$$

& $\phi'(x') = \phi(x) - \frac{i}{2}\omega_{\rho\nu} S^{\rho\nu}\phi \Rightarrow \underline{\frac{\delta\phi}{\delta\omega_{\rho\nu}} = -i S^{\rho\nu}\phi}$

* $\phi(x)$ is a multi-component field here; $S^{\rho\nu}$ is a matrix acting in this space.

$$\phi'(x') = \phi(x') - \sum_{\rho<\nu} \overbrace{\omega_{\rho\nu}}\, (g^{\mu\rho}x^\nu - g^{\mu\nu}x^\rho)\partial_\mu\phi - \frac{i}{2}\sum_{\rho\neq\nu}\omega_{\rho\nu} S^{\rho\nu}\phi$$

$$= \phi(x') - \frac{i}{2}\sum_{\rho\neq\nu}\omega_{\rho\nu} L^{\rho\nu}\phi$$

$$\Rightarrow \boxed{-i L^{\rho\nu}\phi = [\, x^\rho\partial^\nu - x^\nu\partial^\rho - i S^{\rho\nu}]\phi}$$

* comment: $L^{\rho\nu} = i(x^\rho\partial^\nu - x^\nu\partial^\rho) + S^{\rho\nu}$. It seems that this definit has a sign difference from $L_z = x_1 P_2 - x_2 P_1$. The reason is that

$$\begin{cases} x' = x\cos\theta - y\sin\theta \\ y' = y\sin\theta + x\cos\theta \end{cases} \Rightarrow \theta \text{ is } -\omega_{12}.$$

plug in $j^{a,\mu} = \left[-\delta^{\mu}{}_{\lambda} \mathcal{L} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\lambda \phi \right] \frac{\delta x^\lambda}{\delta \omega_a} - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \frac{\delta \phi}{\delta \omega_a}$

set $a = \nu\rho$

$$\Rightarrow j^{\mu\nu\rho} = \left[-\delta^{\mu}{}_{\lambda} \mathcal{L} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\lambda \phi \right] \left[g^{\lambda\nu} x^\rho - g^{\lambda\rho} x^\nu \right] + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} i S^{\nu\rho} \phi$$

$$= \left[-g^{\mu\nu} \mathcal{L} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial^\nu \phi \right] x^\rho - \left[-g^{\mu\rho} \mathcal{L} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial^\rho \phi \right] x^\nu + i \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} S^{\nu\rho} \phi$$

$$\boxed{j^{\mu\nu\rho} = \left(T_c^{\mu\nu} x^\rho - T_c^{\mu\rho} x^\nu \right) + i \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} S^{\nu\rho} \phi}$$

参见 BYB, T_c can be augmented such that

$$\boxed{j^{\mu\nu\rho} = T^{\mu\nu} x^\rho - T^{\mu\rho} x^\nu}$$

where $T^{\mu\nu}$ is Belinfante factor.

Now use $\partial_\mu \langle j^{\mu\nu\rho}(x) \, O \rangle = -i \sum_{i=1}^{N} \delta(x - x_i) \langle \phi_1(x_1) \cdots \hat{L}^{\nu\rho} \phi_i(x_i) \cdots \phi_n(x_n) \rangle$

$$\Rightarrow \partial_\mu \left\langle \left(T^{\mu\nu} x^\rho - T^{\mu\rho} x^\nu \right) O \right\rangle = \sum_{i=1}^{N} \delta(x - x_i) \left[-x^\rho \partial_i^\nu + x^\nu \partial_i^\rho - i S^{\nu\rho} \right] \langle O \rangle$$

plug in $\partial_\mu \langle T^{\mu\nu} O \rangle x^\rho - \partial_\mu \langle T^{\mu\rho} O \rangle x^\nu = - \sum_{i=1}^{N} \delta(x - x_i) \left[\partial_i^\nu \langle O \rangle x^\rho + \partial_i^\rho \langle O \rangle x^\nu \right]$

$$\Rightarrow \langle T^{\mu\nu} O \rangle \partial_\mu x^\rho - \langle T^{\mu\rho} O \rangle \partial_\mu x^\nu = -i \sum_{i=1}^{N} \delta(x - x_i) \, S_i^{\nu\rho} \langle O \rangle$$

$$\Rightarrow \boxed{\left\langle \left(T^{\mu\nu}(x) - T^{\nu\mu}(x) \right) O \right\rangle = -i \sum_{i=1}^{N} \delta(x - x_i) \, S_i^{\nu\mu} \langle O(x_1 \cdots x_n) \rangle}$$

Scale invariance

$$x'^{\mu} = x^{\mu} + \lambda x^{\mu}$$

$$\boxed{\frac{\delta x^{\mu}}{\delta \lambda} = x^{\mu}} \qquad \phi'(x') = (1+\lambda)^{-\Delta}\phi(x)$$

$$\phi'(x') = (1-\lambda\Delta)\phi(x) \Rightarrow \phi'(x') = \phi(x) - \lambda\Delta\,\phi(x')$$

$$= \phi((1-\lambda)x') - \lambda\Delta\,\phi(x')$$

$$= \phi(x') - \lambda x^{\mu}\partial_{\mu}\phi(x') - \lambda\Delta\,\phi(x')$$

$$\Rightarrow \boxed{(-x^{\mu}\partial_{\mu} - \Delta)\phi = -iG\phi, \text{ and } \frac{\delta\phi}{\delta\lambda} = -\Delta\phi}$$

$$\Rightarrow j^{\mu} = \left[-\delta^{\mu}_{\nu}\mathcal{L} + \frac{\partial \mathcal{L}}{\partial(\partial_{\mu}\phi)}\partial_{\nu}\phi\right]\frac{\delta x^{\nu}}{\delta\lambda} - \frac{\partial \mathcal{L}}{\partial(\partial_{\mu}\phi)}\frac{\delta\phi}{\delta\lambda}$$

$$= \left[-\delta^{\mu}_{\nu}\mathcal{L} + \frac{\partial \mathcal{L}}{\partial(\partial_{\mu}\phi)}\partial_{\nu}\phi\right]x^{\nu} - \frac{\partial \mathcal{L}}{\partial(\partial_{\mu}\phi)}(-\Delta)\phi$$

$$= T_{c}{}^{\mu}{}_{\nu}\,x^{\nu} + \Delta\frac{\partial \mathcal{L}}{\partial(\partial_{\mu}\phi)}\phi$$

See BYB P102 Eq 4.42, $T_{c}{}^{\mu}{}_{\cdot\nu}$ can be further made dimensionless based on the Belinfante tensor, and we denote it as $T^{\mu}{}_{\nu}$.

$$\boxed{j^{\mu} = T^{\mu}{}_{\nu}\,x^{\nu}}$$ I will not check it here

then $\partial_{\mu}\langle j^{\mu}(x)\,O\rangle = -i\sum_{i}\delta(x-x_i)\langle\phi(x_1)\cdots\hat{G}\phi(x_i)\cdots\rangle$

$$= -\sum_{i}\delta(x-x_i)\left[x_i^{\mu}\partial_{i,\mu} + \Delta_i\right]\langle O\rangle$$

Combine $\partial_{\mu}\langle T^{\mu}{}_{\nu}\,O\rangle = -\sum_{i}\delta(x-x_i)\,\partial_{\nu,i}\langle O\rangle$

$$\Rightarrow \boxed{\langle T^{\mu}{}_{\mu}(x)\,O\rangle = -\sum_{i}\delta(x-x_i)\,\Delta_i\langle O\rangle}$$

how about the conformal transf (SCT) $\boxed{\dfrac{x'^\mu}{x'^2} = \dfrac{x^\mu}{x^2} - b^\mu}$

$$x'^\mu = \frac{x^\mu - b^\mu x^2}{1 - 2b^\mu x_\mu + b^2 x^\mu x_\nu}$$ under infinitesmall b^μ

$$\Rightarrow \quad x'^\mu = x^\mu + 2(x\cdot b)x^\mu - b^\mu x^2$$
$$= x^\mu + 2b^\nu x_\nu x^\mu - b^\mu x^2$$
$$\Rightarrow \boxed{\frac{\delta x^\mu}{\delta b^\nu} = 2x^\mu x_\nu - x^2 \delta^\mu{}_\nu}$$

how $\phi'(x')$ is related to $\phi(x)$?

$$\phi'(x') = \phi(x) - \frac{i}{2}\omega_{\rho\nu}(x) S^{\rho\nu}\phi$$
$$- \lambda(x)\Delta\phi$$

Spec conformal transf: λ and $\omega_{\rho\nu}$ are spacially dependent.

we can read it from $x'^\mu = x^\mu + 2b^\nu x_\nu x^\mu - b^\mu x^2$

with $\lambda(x) = 2b^\nu x_\nu$, $\quad \omega^\mu{}_\nu = 2b_\nu x^\mu$

$$\Rightarrow \omega_{\rho\nu} = 2x_\rho b_\nu$$

$$\Rightarrow \phi'(x') = \phi(x) - 2b^\nu x_\nu \Delta\phi - i\, x_\rho b_\nu S^{\rho\nu}\phi$$
$$= \phi(x' - 2b^\nu x_\nu x^\mu + b^\mu x^2) - 2b^\nu x_\nu \Delta\phi - i x_\rho b_\nu S^{\rho\nu}\phi$$
$$= \phi(x') + (b^\mu x^2 - 2b^\nu x_\nu x^\mu)\partial_\mu\phi - 2b^\nu x_\nu \Delta\phi - i x_\rho b_\nu S^{\rho\nu}\phi$$

Compare with $\phi'(x') = \phi(x') - i\, b^\mu K_\mu \phi(x')$

$$\Rightarrow \boxed{K_\mu = (i x^2 \partial_\mu - 2i x_\mu x^\nu \partial_\nu) - 2i x^\mu \Delta + x^\rho S_{\rho\mu}}$$

Spec. conformal gerenrater

$$\boxed{\frac{\delta\Phi}{\delta b^\mu} = (-2x^\mu\Delta + i x^\rho S_{\rho\mu})\phi}$$

The conformal current $j^{\mu\nu}$

$$j^{\mu}{}_{\nu} = [-\delta^{\mu}{}_{\nu} \mathcal{L} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\lambda \phi] \frac{\delta x^\lambda}{\delta b^\nu} - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \frac{\delta \phi}{\delta b^\nu}$$

$$= [-\delta^{\mu}{}_{\nu} \mathcal{L} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\lambda \phi][2x^\lambda x_\nu - x^2 \delta^\lambda{}_\nu] - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} (-2x^\nu \Delta + i x^\rho S_{\rho\nu})\phi$$

$$= T_c{}^{\mu}{}_{\lambda}\, 2x^\lambda x_\nu - x^2 T_c{}^{\mu}{}_{\nu} - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} (-2x^\nu \Delta + i x^\rho S_{\rho\nu})\phi$$

we guess ~~that~~ it can be formulated as

$$\boxed{j^{\mu\nu} = T^{\mu\lambda}\, 2x_\lambda x^\nu - T^{\mu\nu} x^2}$$

$$\partial_\mu \langle j^{\mu\nu}(x)\, O \rangle = \partial_\mu \langle (2x_\lambda x^\nu T^{\mu\lambda}_{(x)} - x^2 T^{\mu\nu}_{(x)}) O \rangle$$

$$= -i \sum_i \delta(x - x_i) \langle \phi_1(x_1) \cdots G_\nu \phi_i(x_i) \cdots \phi_n(x_n) \rangle$$

$$-i G_\nu \phi_i(x_i) = (x^2 \partial_\mu - 2x_\nu x^\lambda \partial_\lambda) - 2x_\nu \Delta - i x^\rho S_{\rho\nu}$$

$$\Rightarrow \partial_\mu \langle (2x_\lambda x^\nu T^{\mu\lambda} - x^2 T^{\mu\nu}) O \rangle$$

$$= \sum_i \delta(x - x_i) \langle \phi_1(x_1) \cdots (x_i^2 \partial_{i\nu} - 2x_{i,\nu} x_i^\lambda \partial_{i\lambda}) \phi_i(x_i) \cdots \phi_n(x_n) \rangle$$

$$+ \sum_i \delta(x - x_i) (-2x_{\nu,i} \Delta_i - i x_i^\rho S_{i,\rho\nu}) \langle O \rangle$$

again plug in $\partial_\mu \langle T^{\mu\nu} O \rangle = - \sum_i \delta(x - x_i)\, \partial_i^\nu \langle O \rangle$

the first term of RHS cancels with LHS

we have $\langle [2\partial_\mu(x_\lambda x^\nu)T^{\mu\lambda} - \partial_\mu(x^2)T^{\mu\nu}] O\rangle$

$$= \sum_i \delta(x-x_i)\,(-2x_{i,\nu}\Delta_i - i x_i^\rho S_{i,\rho\nu})\langle O\rangle$$

$$\text{LHS} = \langle [2(\delta_\mu{}^\lambda x_\nu + \delta_\mu{}^\nu x_\lambda)T^{\mu\lambda} - 2x_\mu T^{\mu\nu}] O\rangle$$

$$= \langle [2x_\nu T^{\mu\mu} + 2T^{\nu\lambda}x_\lambda - 2x_\mu T^{\mu\nu}] O\rangle$$

$$= 2x^\nu \langle T^\mu{}_\mu O\rangle + 2x_\mu\langle [T^{\nu\mu} - T^{\mu\nu}] O\rangle$$

plug in

$$\langle T^\mu{}_\mu O\rangle = -\sum_i \delta(x-x_i)\Delta_i\langle O\rangle$$

and $\langle (T^{\mu\nu} - T^{\nu\mu}) O\rangle = -i\sum_i \delta(x-x_i) S_i^{\nu\mu}\langle O\rangle$

$$\Rightarrow \text{LHS} = -2\sum_i \delta(x-x_i)\, x_i^\nu \Delta_i\langle O\rangle$$

$$+ 2i x_\mu \sum \delta(x-x_i) S_i^{\nu\mu}\langle O\rangle \quad -2ix^\rho\sum_i \delta(x-x_i) S_i^{\rho\nu}\langle O\rangle$$

(it seems that there's a facter of 2 mismatch, which will be checked later)

but the special conformal ~~transf~~ doesn't bring new conservation law!

Now we summarize Ward identities

$$\boxed{\begin{aligned} &\partial_\mu \langle T^\mu{}_\nu(x)\, O\rangle = -\sum_i \delta(x-x_i)\, \partial_{i,\nu} \langle O\rangle \\ &\langle (T^{\mu\nu}(x) - T^{\nu\mu}(x))\, O\rangle = -i \sum_i \delta(x-x_i)\, S_i^{\nu\mu} \langle O\rangle \\ &\langle T^\mu{}_\mu(x)\, O\rangle = -\sum_i \delta(x-x_i)\, \Delta_i \langle O\rangle \end{aligned}}$$

① ② ③

in 2D ② simplifies →

$$\varepsilon_{\mu\nu} \langle T^{\mu\nu}(x)\, O\rangle = -i \sum_i S_i\, \delta(x-x_i) \langle O\rangle$$

where S_i is the spin of the field ϕ_i.

another result: from $\frac{\partial}{\partial x^\mu} \langle j^\mu_a\, O\rangle = -i \sum_{i=1}^n \delta(x-x_i) \langle \phi_1(x_1) \cdots G_a \phi_i(x_i) \cdots \phi_n(x_n)\rangle$

$$\int_\Sigma dS_\mu \langle j^\mu_a\, O\rangle = \int d^dx\, -i \sum_{i=1}^n \delta(x-x_i) \langle \phi_1(x_1) \cdots G_a \phi_i(x_i) \cdots\rangle = \sum_{i=1}^n \langle \Phi(x_1) \cdots G_a \phi_i(x_i) \cdots \phi_n\rangle$$

Since it's a symmetry operation:

$$\boxed{\sum_{i=1}^n \langle \phi_1(x_1) \cdots G_a \phi_i(x_i) \cdots \phi_n(x_n)\rangle = 0}$$

$$\Rightarrow \int_\Sigma dS_\mu \langle j^\mu_a\, O\rangle = 0$$