Lect 17 Discrete symmetry: time-reversal and parity (TR)

1. Wigner's theorem:

Generally speaking, for a transformation \( R \) (not necessarily linear), if it does not change the magnitude of the inner product between two arbitrary state vectors \( |\Psi\rangle \) and \( |\Phi\rangle \), i.e., \( |\langle \Psi | \Phi \rangle| = |\langle R\Psi | R\Phi \rangle| \), then \( R \) is either a unitary transformation, or, an anti-unitary transformation. (We will omit the proof). For continuous transformation \( R \) is unitary (Why?).

Anti-unitary transformation \( R \) means that: for a super-position between \( |\phi_1\rangle, |\phi_2\rangle \),

\[
R \left( c_1 |\phi_1\rangle + c_2 |\phi_2\rangle \right) = c^* R |\phi_1\rangle + c^* R |\phi_2\rangle,
\]

or \( R c = c^* R \).

Usually anti-unitary transformation can be expressed as \( R = U K \), where \( U \) is an usual unitary transformation, and \( K \) is anti-unitary satisfying \( K K = 1 \). In the coordinate representation, we choose \( K \) as complex conjugation.

\[
\langle \vec{r} | K | \Psi \rangle = \langle \vec{r} | \Psi \rangle^*\]

Ex: please check that \( R^{-1} = KU^+ = KU^{-1} \), and we can evaluate

\[
\langle R\Psi | R\Phi \rangle = \langle uK\Psi | uK\Phi \rangle = \langle K\Psi | K\Phi \rangle = \text{det} \langle K\Psi | K\Phi \rangle = \text{det} \langle \Psi | \Phi \rangle = \text{det} \langle \Phi | \Psi \rangle
\]

\[
= \text{det} \langle \Phi | \Psi \rangle^* = \text{det} \langle \Phi | \Psi \rangle = \text{det} \langle \Phi | \Psi \rangle^* = \text{det} \langle \Psi | \Phi \rangle = \text{det} \langle \Phi | \Psi \rangle
\]

\[
\Rightarrow \langle R\Psi | R\Phi \rangle = \langle \Phi | \Psi \rangle^*
\]

Ex: prove that \( \langle R^{-1} \Psi | R^{-1} \Phi \rangle = \langle \Phi | \Psi \rangle = \langle \Psi | \Phi \rangle^* \).
For any states $|\psi\rangle$ and $|\psi'\rangle$, and operator $O$

$$\langle R\psi | 0 | R^{-1}R|\psi'\rangle = \langle \psi R^{-1}O R|\psi'\rangle^*$$

If $|\psi\rangle = |\psi'\rangle$, and $O$ is an Hermitian operator $\Rightarrow \langle R\psi | 0 | R|\psi\rangle \geq 0$

$\Rightarrow \langle R\psi | 0 | R|\psi\rangle = \langle \psi R^{-1}O R|\psi\rangle$.

### §2. TR transformation

Consider a state vector $|\psi\rangle$, and its TR counterpart $|\psi^T\rangle = T|\psi\rangle$

or, equivalently $|\psi\rangle = T^{-1}|\psi^T\rangle$, We assume $T$ and $T^{-1}$ satisfy Wigner theorem. Now we need to determine $T$ is unitary or anti-unitary.

We need correspondence principle.

In order to agree with classical mechanics, we need maintain

$$\begin{align*}
\langle \psi^T | \vec{p}^T | \psi^T \rangle &= \langle \psi | \vec{p} | \psi \rangle, \\
\langle \psi^T | \vec{p} | \psi^T \rangle &= -\langle \psi | \vec{p} | \psi \rangle
\end{align*}$$

\[
\begin{bmatrix}
\langle \psi^T | \vec{p}^T | \psi^T \rangle \\
\langle \psi^T | \vec{L}^T | \psi^T \rangle
\end{bmatrix} = 
\begin{bmatrix}
\langle \psi | \vec{p} | \psi \rangle \\
\langle \psi | \vec{L} | \psi \rangle
\end{bmatrix}
\]

$T^{-1}\vec{r}T$ is a linear operator since the product of two anti-linear operators is a linear operator. The above relation is valid for any state vector $|\psi\rangle$. It's easy to show for two arbitrary state vectors

$$\langle \psi_1 | T^{-1}\vec{r}T | \psi_2 \rangle = \langle \psi_1 | \vec{r} | \psi_2 \rangle$$

such that $T^{-1}\vec{r}T = \vec{r}$.

Proof: take $|\psi\rangle = |\psi_1\rangle + |\psi_2\rangle$,
\[
\left( |\psi_1\rangle + |\psi_2\rangle \right) (T^r T) \left( |\psi_2\rangle + i |\psi_1\rangle \right) = \left( |\psi_1\rangle + i |\psi_2\rangle \right) (T) \left( |\psi_2\rangle + i |\psi_1\rangle \right)
\]

\[
\Rightarrow \left\{ \begin{array}{l}
\langle \psi_1 | T^r T | \psi_2 \rangle + \langle \psi_2 | T^r T | \psi_1 \rangle = \langle \psi_1 | \hat{T}^r \psi_2 \rangle + \langle \psi_2 | \hat{T}^r \psi_1 \rangle \\
\text{if we take } |\psi \rangle = 1 |\psi_1 \rangle + i |\psi_2 \rangle \Rightarrow \\
\langle \psi_1 | T^r T | \psi_2 \rangle - \langle \psi_2 | T^r T | \psi_1 \rangle = \langle \psi_1 | \hat{T}^r \psi_2 \rangle - \langle \psi_2 | \hat{T}^r \psi_1 \rangle
\end{array} \right.
\]

\[
\Rightarrow \left| \psi_1 \right| T^r T \left| \psi_2 \right> = \langle \psi_1 | \hat{T}^r \psi_2 \rangle.
\]

Similarly, we should have

In order to be consistent with these relation, \( T \) has to be anti-unitary.

Check the commutation relation - \([x, p] = i \hbar\), how does it change under \( T \)?

\[
T [x, p] T^{-1} = T i \hbar T^{-1}
\]

\[
(T x T^{-1})(T p T^{-1}) - (T p T^{-1})(T x T^{-1}) = -(xp - px) = -i \hbar
\]

\[
\Rightarrow \quad T i T^{-1} = -i
\]

Ex: From \([L_i, L_j] = i \epsilon_{ijk} L_k\), derive that \( T i T^{-1} = -i \).

\[S3. \quad T^2 = ?\]

Naively, we would expect that after TR transformation twice, the system comes back to itself, thus \( T^2 = 1 \). But we will see two possibilities.
First, $T^2$ is a constant.

Proof: we have $\vec{T} \cdot \vec{T}^{-1} = \vec{1} \Rightarrow \vec{T}^2 \cdot \vec{T}^{-1} = \vec{1} \Rightarrow \vec{T}^2 = \vec{T}^2 \cdot \vec{T}^{-1} = \vec{1} \Rightarrow \vec{T}^2 = \vec{T}^2$ and similarly $T^2 \vec{\ell} = \vec{\ell} \vec{T}^2$, $T^2 \vec{s} = \vec{s} \vec{T}^2$, $T^2 i = iT^2$.

For any operator $F(r, p, s, i)$, we have $T^2 F(r, p, s, i) = F(r, p, s, i)T^2 \Rightarrow T^2$ is a constant.

Then what's its value? Answer: $T^4 = 1$, and thus $T^2 = \pm 1$.

Proof: $T^4 = T(T^2)T = (T^2)^*T^2 = T^2(T^2)^*$.

For any two state vectors $|\psi\rangle$ and $|\phi\rangle$, remember $T$ is anti-unitary, and $T^2$ is a complex constant $\Rightarrow$

$$<T\psi|T\phi> = <\psi|\phi>^*, \quad <T^2\psi|T^2\phi> = <T\psi|T\phi>^* = <\psi|\phi>$$

$$<T^2\psi|T^2\phi> = (T^2)^*T^2 <\psi|\phi> = <\psi|\phi> \Rightarrow (T^2)^*T^2 = T^4 = 1$$

§4 The case of $T^2 = 1$.

For single component system, we can simply define $\psi^T(r) = \psi^*(r)$, or $<r|T\psi> = <r|\psi>^*$. Please check that it satisfies:

$$\int d^3r (\psi^T(r))^* \begin{bmatrix} \vec{p} \\ \vec{r} \end{bmatrix} \psi^*(r) = \int d^3r \psi^*(r) \begin{bmatrix} \vec{p} \\ -\vec{r} \end{bmatrix} \psi(r).$$
Is there an example of $H$, that violates TR symmetry?

**Ex:**

check: $H = \frac{(p - eA)^2}{2m}$, what's $H^T = TH T^{-1} =$?

§5 The case of $T^2 = -1$, and Kramer degeneracy.

Let's consider a system with spin. The rotation matrix

$$D(q) = e^{-i\hat{f}\hat{n}\theta}$$

Let's consider the rotation operation and TR

$$D^\dagger(g(y, \pi)) J_z D(g(y, \pi)) = -J_z$$

$$T^{-1} J_z T = -J_z$$

$$T^{-1} D^\dagger(g(y, \pi)) J_z D(g(y, \pi)) T = J_z$$

$$\Rightarrow [D(g(y, \pi)) T] J_z = [D(g(y, \pi)) T] J_z$$

Consider an $J_z$ eigenstate, $|jm\rangle$, then $D(g(y, \pi)) T |jm\rangle$ must be the same as $|m\rangle$ up to a complex constant, because

$$J_z \left( D(g(y, \pi)) T |jm\rangle \right) = D(g(y, \pi)) T J_z |jm\rangle = m \left( D(g(y, \pi)) T |jm\rangle \right)$$

$$\Rightarrow D(g(y, \pi)) T |jm\rangle = c |jm\rangle .$$

Then $(D(g(y, \pi)) T)^2 |jm\rangle = (D(g(y, \pi)) T) c |jm\rangle = c^* c |jm\rangle ,$

$$\langle D(g(y, \pi)) T jm | D(g(y, \pi)) T jm \rangle = \langle T jm | T jm \rangle = \langle jm | jm \rangle \Rightarrow c^* c = 1$$
Thus \((D(g,y,\pi) T)^2 = 1\), or \(T D(g,y,\pi) T D(g,y,\pi) = 1\)

For \(e^{-i J \cdot \hat{n} \theta} T^{-1} = e^{-(i)(-J) \cdot \hat{n} \theta} = e^{-i J \cdot \hat{n} \theta}\)

\(\Rightarrow T D(g) = D(g) T\)

\(\Rightarrow T^2 D(g,y,\pi) = 1\) or \(T^2 D(g,y,2\pi) = 1\)

But rotation around \(y\)-axis at \(2\pi\)-angle, should it just an identity transformation? Not quite

\(D(g,y,2\pi) = \begin{cases} 1 & \text{if } j \text{ integer} \\ -1 & \text{if } j \text{ half-integer} \end{cases}\)

Proof:

\(D(g,y,2\pi) = D(g(x,\pi)) D(g(z,2\pi)) D(g^{-1}(x,\pi))\)

\(g(x,\pi)\) rotation rotates \(y\)-axis into \(z\)-axis

\(D(g(z,2\pi)) = e^{-i J_z 2\pi} = \begin{cases} 1 & \text{if } J_z \text{ integer} \\ -1 & \text{if } J_z \text{ half-integer} \end{cases}\)

\(\Rightarrow D(g,y,2\pi) = D(g,z,2\pi) = \begin{cases} 1 & \text{if } j \text{ integer} \\ -1 & \text{if } j \text{ half-integer} \end{cases}\)

\(T^2 = \begin{cases} 1 & \text{for } j \text{ integer} \\ -1 & \text{for } j \text{ half integer} \end{cases} \rightarrow\) orthogonal class, symplectic class.
(1) For spin-$\frac{1}{2}$ case, a convenient choice is $T = -i\sigma_y \ K = (1, -1)^T$

- $T |\uparrow\rangle = |\downarrow\rangle$
- $T |\downarrow\rangle = -|\uparrow\rangle$

and

$T (C_1 |\uparrow\rangle + C_2 |\downarrow\rangle) = C_1^* |\downarrow\rangle - C_2^* |\uparrow\rangle$.

$k$ is the complex conjugate on complex coefficient

or

$T (C_1) = (-C_2^*)^T, \quad T^2 (C_1) = -(C_1)$. 

(2) For a general case, we can define $T = R K$.

- if $j$ half integer, $2j+1$ is even, we may choose $R = (1, -1)^T$, and $R^2 = -1$, and $T^2 = -1$.

- if $j$ is integer, $2j+1$ is odd, we choose $R = (1, -1, \cdots, m=0)$, such that $R^2 = T^2 = 1$.

in this case, $T |lm\rangle = (-)^m |l-m\rangle$, and

$$\langle \hat{n} | T |lm\rangle = Y^*_l m(\theta, \varphi) = (-)^m \langle \hat{n} | l-m\rangle = (-)^m Y^*_{l-m}(\theta, \varphi).$$

Consistent with $Y^*_l m(\theta, \varphi) = (-)^m Y_{l-m}(\theta, \varphi)$

§ Kramer degeneracy.

if $T^2 = -1$, then for any state $|\psi\rangle$, with $H |\psi\rangle = E |\psi\rangle$,

then $H(T |\psi\rangle) = T H |\psi\rangle = E(T |\psi\rangle)$, thus $T |\psi\rangle$ is also an eigenstate with the same energy.
On the other hand

\[ \langle \psi | T \psi \rangle = \langle T \psi | T^2 \psi \rangle^* = - \langle T \psi | \psi \rangle^* = - \langle \psi | T \psi \rangle \]

\[ \Rightarrow \langle \psi | T \psi \rangle = 0. \text{ thus } T \psi \text{ is an other state, and there's at least 2-fold degeneracy.} \]

Ex: if \( T^2 = 1 \), is there always an energy level degeneracy?

\section{Parity transformation}

Consider a state vector \( |\psi\rangle \), after parity transformation \( P \), we have

\[ |\psi^p\rangle = P |\psi\rangle \text{ or } |\psi\rangle = P^{-1} |\psi^p\rangle. \]

Again we assume \( P \) satisfies Wigner theorem. Again we use correspondence principle, and arrive at

\[ \langle \psi^p | \vec{r} | \psi^p \rangle = - \langle \psi | \vec{r} | \psi \rangle, \quad \langle \psi^p | \vec{p} | \psi^p \rangle = - \langle \psi | \vec{p} | \psi \rangle \]

\[ \langle \psi^p | \vec{L} | \psi^p \rangle = \langle \psi | \vec{L} | \psi \rangle, \text{ and also } \langle \psi^p | \vec{S} | \psi^p \rangle = - \langle \psi | \vec{S} | \psi \rangle \]

\[ \Rightarrow P \vec{r} \vec{p} = - \vec{r}, P \vec{p} \vec{p} = - \vec{p}, \text{ and } P \vec{L} \vec{p} = \vec{L}, P \vec{S} \vec{p} = \vec{S} \]

check \( [x, p] = i \hbar \Rightarrow P \vec{x} \vec{p} = [-\vec{x}, -\vec{p}] = i \hbar = P (i \hbar) \vec{p} \]

\[ \Rightarrow P i = i P \Rightarrow P \text{ is an unitary transformation.} \]

\[ \Rightarrow \]

\[ \text{Similarly, we can also prove that } P^2 \text{ is a constant, and } P^2 (i P)^* = 1. \]

\[ \text{Ex: } \]

without loss of generality, we choose

\[ \Rightarrow P^2 = e^{i \theta} \text{ up to phase factor. } P^2 = 1. \]
For single component system, we simply set $\psi^p(\vec{r}) = \psi(-\vec{r})$.

We can easily check this definition satisfy the above requirement!

For the time-dependent case, we can define

\[
\begin{align*}
\psi^T(x, t) &= \psi^*(x, -t) \\
\psi^p(x, t) &= \psi(-x, t).
\end{align*}
\]

Ex: verify for momentum eigenstate $\psi_p(x, t) = e^{i p x - i \omega t}$, what are $\psi^T_p(x, t)$ and $\psi^p_p(x, t)$? How about angular momentum eigenstates $\psi_m(x, t) = e^{i m \varphi - i \omega t}$?

§ Parity broken in weak interactions. C. N. Yang and T. D. Lee (Theory proposal)

(left handed)

\[
\begin{pmatrix}
e_L \\
\nu_L
\end{pmatrix}, \quad e_R, \quad \text{there's no } \nu_R.
\]

C. S. Wu's experiments.
\section{Parity eigenstates}

- If $[H, \hat{P}] = 0$, then we can find common eigenstates of $H$ and $\hat{P}$.

  For example: 1D harmonic oscillator $\hat{P}^\dagger \hat{P} = H$. It's energy eigenfunctions are:
  \[ \psi_n(x) = (-1)^n \psi_0(x), \quad \text{even for } n=0, 2, 4 \]
  \[ \psi_n(x) = (-1)^n \psi_0(x), \quad \text{odd for } n=1, 3, 5 \]

- Orbital angular momentum eigenstates $Y_{lm}(\hat{n})$
  \[ Y_{lm}(-\hat{n}) = (-1)^l Y_{lm}(\hat{n}) \]

- Selection rule $\Delta l = \pm 1$
  \[ \langle nlm | \Delta l | n'l'm' \rangle \neq 0, \quad \text{only for } l' = l \pm 1. \]

\section{The relation between degeneracy and symmetry}

For a Hamiltonian, all its symmetry operations together form a group $\{R\}$.

- If a state $|\psi\rangle$ is an eigenstate, then all the states $R|\psi\rangle$
  \[ H(R|\psi\rangle) = R(H|\psi\rangle) = E|\psi\rangle \]
form a subspace. This subspace supports a representation for group $\{R\}$ of the symmetry.

For example:
\begin{enumerate}
  \item For 3D rotation symmetry, all the states $\psi_{lm} = R_{\theta \phi} Y_{lm}(\theta, \phi)$ with $m = -l, \ldots, l$, form a $l$-fold degeneracy.
  \item But for 1D harmonic oscillator, the parity symmetry doesn't bring degeneracy.
\end{enumerate}
whether degeneracy appears or not depends on the nature of the sym


group. If the group is Abelian, i.e. every symmetry operation

by commute with other, we do not expect degeneracy. It's because Abelian

group usually only support 1d representation.

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Example: 2\text{Parity group} \{I, \pi\}. No degeneracy in 1D harmonic oscillator

If we apply \( \pi \psi_n(r) = \pm \psi_n(r) \), no new states appear.

\( H = -B\cdot \hat{S}_z \), uni-axial rotation symmetry.

\( \text{SO}(2) \) group: \( \{ e^{-i\hat{S}_z \theta} \} \). Again for its eigenstates \( |s s_z\rangle \)

we have \( \hat{S}_z |s s_z\rangle = s_z |s s_z\rangle \), no more states.

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Energy level degeneracy are usually associated with non-Abelian sym


group. Only non-Abelian group supports multi-dimensional representations.

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Example: 2D rotator \( H = \frac{\hat{l}_z^2}{2I} \).

Each level except the ground state is 2-fold symmetric. \( \psi_{\pm m} = e^{\pm i m \theta} \).

The symmetry group is \( \text{O}(2) \) not \( \text{SO}(2) \): \( \{ e^{-i\hat{l}_z \theta} \} \cup \{ \Pi_x e^{-i\hat{l}_z \theta} \} \),

where \( \Pi_x \) is the reflection with respect to x-axis. It's easy to check

\( \Pi_x e^{-i\hat{l}_z \theta} \Pi_x = e^{i\hat{l}_z \theta} \), and thus \( \text{O}(2) \) is non-abelian!

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For its eigenstates \( \psi_{\pm m} \), we have \( \hat{l}_z \psi_{\pm m} = \pm m \psi_{\pm m} \), and

\( \Pi_x \psi_{-m} = \psi_m \) and \( \Pi_x \psi_{m} = \psi_{-m} \).