Lect 15: The D-matrix

Theorem: The Hilbert space spanned by $|jm\rangle$ ($m=-j, \ldots, j$) is rotationally invariant and irreducible.

Proof: We denote such a space as $L^j$. Any vector in such a space can be expanded as $|\psi_j\rangle = \sum_m a_m |jm\rangle$. For any rotation $g(n,\theta)$, the associated rotation operator $D_g = e^{i\hat{J}_n\theta}$, we have

$$D_g |\psi_j\rangle = \sum_m a_m e^{i\hat{J}_n\theta} |jm\rangle.$$

$e^{i\hat{J}_n\theta} = \sum_{n=0}^\infty \frac{\hat{J}_n^n}{n!} \langle \hat{J}_n \rangle^n$ is a function of $J_2, J_\pm$, and we know $J_2, J_\pm$ do not change the value of $j$. Thus $e^{i\hat{J}_n\theta} |jm\rangle$ remains inside $L^j$, and $L^j$ is invariant under rotation.

The proof of the irreducibility is more tricky. It means for any state in $L^j$, by acting $J_\pm$ successively, we can arrive at $2j+1$ linearly independent states. Thus there's no a smaller subspace inside $L^j$. We will not give a rigorous proof here.

§ Representation of rotation group

Any rotation $g(\hat{R},\theta)$ can be represented as a 3x3 orthogonal matrix, and mathematically called $SO(3)$ group, or, isomorphically, $SU(2)$. The only difference between $SO(3)$ and $SU(2)$ is that $SU(2)$ includes
both half integer angular momentum, and SO(3) only includes integer.

Loosely speaking, we often do not distinguish this difference. Check Sakurai Sect 3.3 for more info.

Quantum mechanically, $g$ is represented by the rotation operator $D(g) = e^{-i \hat{J} \hat{n} \theta}$, In the space of $L^j$, defined above, $D(g)$ is further represented by a $(2j+1) \times (2j+1)$ matrix as

$$D_{m'm}^j(g) = \langle jm' | e^{-i \hat{J} \hat{n} \theta} | jm \rangle$$

The correspondence between $g \rightarrow D(g) \rightarrow D_{m'm}^j(g)$ follows the product of matrix:

$$g \rightarrow D(g) \rightarrow D_{m'm}^j(g)$$

$$g = g_1 g_2 \rightarrow D(g) = D(g_1) D(g_2) \rightarrow D_{m'm}^j(g) = \sum_{m''} D_{m''m}^j(g_1) D_{m''m}^j(g_2)$$

rotation operation $\rightarrow$ QM rotation operator $\rightarrow$ Rotation $D$-matrix in the space $L^j$.

Ex: please prove that $D_{m'm}^j(g)$ is a unitary matrix.

§ Calculation of $D_{m'm}^j(g)$.

The parameterization of $g(\hat{n}, \theta)$ is not convenient for later use. We use the Eulerian angles, which connects body frame and lab frame.
nicely. Now we define \( g(\alpha, \beta, \gamma) \) as a three-step rotation:

1. Rotate around \( z \)-axis at \( \alpha \)-angle, then \((x', y', z') \xrightarrow{g(\hat{z}, \alpha)} (x', y', z' = \hat{z})\)

2. Rotate around \( y' \)-axis with \( \beta \)-angle, then \((x', y', z') \xrightarrow{g(y', \beta)} (x'', y'' = \hat{y}', z'')\)
   (NOT \( y \)-axis, but the new \( y \)-axis)

3. Rotate around \( z'' \)-axis, then \((x'', y'', z'') \xrightarrow{g(\hat{z}, \alpha)} (x''', y''', z''' = \hat{z}'')\)
   (the new \( z \)-axis)

Thus \( g(\alpha, \beta, \gamma) = g(\hat{z}', \alpha) \cdot g(\hat{y}', \beta) \cdot g(\hat{z}, \alpha) \)

Let's check the relation \( g(\hat{z}'', \gamma) \) and \( g(\hat{z}, \alpha) \).

2. First, rotation \( g(\hat{y}', \beta) \) apply on \( \hat{z} \rightarrow \hat{z}'' \), i.e. \( \hat{z}'' = g(\hat{y}', \beta) \hat{z} \).

From the rotation theory, we have \( g(\hat{z}'', \gamma) = g(\hat{y}', \beta) \cdot g(\hat{z}, \alpha) \cdot g^{-1}(\hat{y}', \beta) \)

Please check! Step 1: apply \( g^{-1}(\hat{y}', \beta) \) such that \( \hat{z}'' \)-axis is restored to \( \hat{z} \).

Step 2: apply rotation around \( \hat{z} \) with same angle \( \gamma \).

Step 3: rotate the system back by \( g(\hat{y}', \beta) \).
\[ g(\hat{z}', \chi) g(y', \beta) = g(y', \beta) g(\hat{z}', \chi) \]

\[ g(\hat{z}', \chi) g(\hat{z}, \alpha) = g(\hat{z}, \alpha) g(y', \beta) \]

Ex2: please prove

Finally, we have

The $3 \times 3$ matrix rep for $g$ is

\[ g(\hat{z}, \alpha) = \begin{pmatrix} \cos \theta & -
\sin \theta & 0 \\
\sin \theta & \cos \theta & 0 \\
0 & 0 & 1 \end{pmatrix} \]

and $g(\hat{y}, \beta) = \begin{pmatrix} \cos \beta & 0 & 
\sin \beta & 0 \\
0 & 1 & 0 \\
-
\sin \beta & 0 & \cos \beta \end{pmatrix}$

Now we map to the $D$-matrix

\[ D(g) = D(g(\hat{z}, \alpha)) D(g(\hat{y}, \beta)) D(g(\hat{z}', \chi)) = e^{iJ_x \alpha} e^{iJ_y \beta} e^{iJ_z \chi} \]

\[ D^j_{m'm'}(\alpha \beta \gamma) = \langle j'm' | e^{iJ_x \alpha} e^{iJ_y \beta} e^{iJ_z \chi} | j'm \rangle \]

\[ = e^{-i m' \alpha - i m \chi} \langle j'm' | e^{iJ_y \beta} | j'm \rangle \]

we define d-matrix:

\[ d^j_{m'm'}(\beta) = \langle j'm' | e^{iJ_y \beta} | j'm \rangle \]

\[ iJ_y \text{ in the representation of } | j'm \rangle, \text{i.e. } i \langle j'm' | J_y | j'm \rangle \text{ are purely real, so does } \langle j'm' | e^{iJ_y \beta} | j'm \rangle. \Rightarrow (d_{m'm'}^j(\beta))^* = d_{m'm'}^j(\beta) \]
Ex 3: prove that 
\[ \alpha_{m'm}(\beta) = \alpha_{m'm}(\beta) \].

§ Expression of \( \alpha_{m'm}(\beta) \).

We use a 2D harmonic oscillator to represent algebra of angular momenta. The creation/annihilation operators \( a, a^\dagger \) can represent \( J_x, J_y, J_z \). The followings follow:

\[ \vec{J} = \frac{1}{2} (a^\dagger a^\dagger) - (a\dagger a) \quad \text{or} \quad J_z = \frac{1}{2} (a^\dagger a - a^\dagger a) \]

Schwinger boson Rep.

\[ J_x = \frac{1}{2} (a^\dagger a_2 + a^\dagger a_1) \]
\[ J_y = \frac{i}{2} (a^\dagger a_2 - a^\dagger a_1) \]

Ex 4: please check the expressions of \( \vec{J} \) in terms of \( a_{1,2}, a_{1,2}^\dagger \) satisfy \( [J_i, J_j] = i \epsilon_{ijk} J_k \), and also \( \vec{J}^2 = (\frac{a^\dagger a_1 + a^\dagger a_2}{2})(\frac{a^\dagger a_1 + a^\dagger a_2}{2} + 1) \).

Next is the map of states. \( |jm\rangle \) corresponds to the state with \( n_1 = a^\dagger a_1 = j + m \), \( n_2 = a^\dagger a_2 = j - m \), i.e.

\[ |jm\rangle = \frac{(a^\dagger)^{j+m}(a^\dagger)^{j-m}}{\sqrt{(j+m)! (j-m)!}} \langle 0| = |n_1 n_2\rangle \]

Ex 5: Check that in the Schwinger boson Rep. we do have

\[ J_z |jm\rangle = m |jm\rangle \]
\[ J_{\pm} |jm\rangle = \sqrt{(j+m)(j+m+1)} |jm\pm 1\rangle \]

← Prove them in the Schwinger boson Rep.
\[ e^{-i\gamma} | jm \rangle = \frac{e^{-i\gamma} (a_1^+)^{j+m} (a_2^+)^{j-m} e^{-i\gamma}}{\sqrt{\Gamma(j+m)! (j-m)!}} | 0 \rangle \]

Define \((a_1^+)^\dagger = e^{i\gamma} (a_1^+)^\dagger e^{-i\gamma}\), we have

\[ e^{-i\gamma} | jm \rangle = \frac{(a_1^+)^{j+m} (a_2^+)^{j-m}}{\sqrt{(j+m)! (j-m)!}} | 0 \rangle \]

Check the note before. \(a_1^+ = a_1^\dagger \cos \frac{\beta}{2} + a_2^\dagger \sin \frac{\beta}{2}\)
\(a_2^+ = -a_1^\dagger \sin \frac{\beta}{2} + a_2^\dagger \cos \frac{\beta}{2}\).

\[ \Rightarrow e^{-i\gamma} | jm \rangle = \frac{(a_1^\dagger \cos \frac{\beta}{2} + a_2^\dagger \sin \frac{\beta}{2})^{j+m} (a_1^\dagger \sin \frac{\beta}{2} + a_2^\dagger \cos \frac{\beta}{2})^{j-m}}{\sqrt{(j+m)! (j-m)!}} | 0 \rangle \]

\[ = \frac{1}{\sqrt{(j+m)! (j-m)!}} \sum_{m'=-j}^{j} \sum_{\sigma} (j+m')(j-m') (a_1^\dagger \cos \frac{\beta}{2})_{m+m'+\sigma} (a_2^\dagger \sin \frac{\beta}{2})_{j-m'-\sigma} (-)^{j-m'-\sigma} (a_1^\dagger \sin \frac{\beta}{2})_m (a_2^\dagger \cos \frac{\beta}{2})_{j-m} | 0 \rangle \]

\[ = \frac{1}{\sqrt{(j+m)! (j-m)!}} \sum_{m'=-j}^{j} \sum_{\sigma} (j+m')(j-m') (a_1^\dagger)^{j+m'} (a_2^\dagger)^{j-m'} (-)^{j-m'-\sigma} (\cos \frac{\beta}{2})_{m+m'+2\sigma} (\sin \frac{\beta}{2})_{2j-2\sigma-m'-m} | 0 \rangle \]

\[ 0 \leq \sigma \leq j-m \]
\[-m-m' \leq \sigma \leq j-m' \]
\[ \Rightarrow \max (0, -(m+m')) \leq \sigma \leq \min (j-m, j-m') \]
\[ a^i_{m'm}(\beta) = \sqrt{\frac{(j+m)!(j-m)!}{(j+m')!(j-m')!}} \sum_{\sigma} \frac{(j+m)(j-m)(-1)^{j-m-\sigma}}{\sigma} (\sin \frac{\beta}{2})^{m+m'+2\sigma} (\sin \frac{\beta}{2})^{2j-2\sigma-m-m'} \]

Important relations without proof: (\(l\) is integer below)

\[ a^l_{m'0}(\beta) = \left[ \frac{(l-m)!}{(l+m)!} \right]^{1/2} P^l_m(\cos \beta) \]

\[ D^l_{m'0}(\alpha, \beta, \gamma) = a^l_{m'0}(\beta) = P^l(\cos \beta) \]

We will prove \( D^l_{m0}(\alpha, \beta, \gamma=0) = \sqrt{\frac{4\pi}{2l+1}} Y^*_{lm}(\theta=\beta, \phi=\alpha) \)

Define \( |\hat{n}\rangle \) as direction eigenket, where \( \hat{n}(\theta, \phi) \) along any solid angle direction. For state \( |lm\rangle \), we have \( \langle \hat{n} | lm \rangle = Y_{lm}(\theta, \phi) \)

\[ |n\rangle = e^{-iJ_z \phi} e^{-iJ_y \theta} |\hat{z}\rangle = \sum_{l'\ell m'} D_{\ell' \ell, m' m}(g(\alpha=\phi, \beta=\theta, \gamma=0)) |l' m' \rangle \langle l m | \hat{z}\rangle \]

\[ \langle lm' | n \rangle = \sum_{\ell' m'} <lm' | D(g) | l m' \rangle <l m' | \hat{z}\rangle = \sum_m D^l_{m' \ell m}(\alpha=\phi, \beta=\theta, \gamma=0) \]

\[ Y^*_{lm'}(\theta, \phi) = \sum_m D^l_{m' \ell m}(\alpha=\phi, \beta=\theta, \gamma=0) Y^*_{lm}(\theta=\theta, \phi \text{ undetermined}) \]

\[ Y_{lm}(\theta=0, \phi) = \sqrt{\frac{2l+1}{4\pi}} P^l_m(\cos \theta)|_{\theta=0} \delta_{m, 0} = \sqrt{\frac{2l+1}{4\pi}} \delta_{m, 0} \]
\[ \gamma_{lm}^{+}(\theta, \phi) = \sqrt{\frac{2l+1}{4\pi}} D_{m', 0}^{l}(\theta = \phi, \beta = \theta, \gamma = 0) \]

or

\[ D_{m, 0}^{l}(\alpha, \beta, \gamma) = \frac{\sqrt{4\pi}}{\sqrt{2l+1}} \gamma_{lm}^{+}(\theta, \phi) \bigg|_{\theta = \beta, \phi = \alpha} \]

Set \( m = 0 \) and use \( \gamma_{l0}(\theta, \varphi) = \sqrt{\frac{2l+1}{4\pi}} P_{l}(\cos \theta) \) ⇒

\[ d_{00}^{l}(\beta) = P_{l}(\cos \theta) \bigg|_{\theta = \beta} \]