Problem 1. (Sakurai 2nd edition 6.4)

Solution:

We quote the formulas in Sakurai's book for the logarithmic derivative and phase shift:

\[ \beta_x = \left( \frac{\partial}{\partial k'} \frac{j_x(k')}{j_x(k')} \right) \bigg|_{k'=k} \]

\[ \tan \delta_x = \frac{K R j_x'(KR) - \beta_x j_x(KR)}{K R j_x''(KR) - \beta_x j_x(KR)} \]

in which \( \frac{\partial}{\partial k'} k' = E - V_0 \), appearing in the logarithmic derivative at \( R = 0 \), while the expression for \( \tan \delta_x \) is a matching of connecting condition between \( R = 0 \) and \( R > 0 \).

Since we're considering low energy scattering (i.e. \( K R < 1 \)), only a small number of partial wave channels contribute. And we can only keep the results to lowest non-Vanishing order of \( K R \), which would be \( (K R)^{2n+4} \) for \( l \)-th partial wave channel known as the so-called threshold behaviour.

The expansions for \( j_{l}(x) \) and \( n_{l}(x) \) around \( x=0 \) are:

\[ j_{l}(x) = \frac{\pi}{2} \sum_{\alpha = 0}^{\infty} \frac{(-1)^\alpha}{n! \Gamma(n+3/2)} \left( \frac{x}{2} \right)^{2n+\alpha} \]

\[ n_{l}(x) = \frac{\pi}{2} \sum_{\alpha = 0}^{\infty} \frac{(-1)^\alpha}{n! \Gamma(n+3/2)} \left( \frac{x}{2} \right)^{2n-\alpha-1} \]

These give:

\[ j_{l}'(x) = \frac{x^l}{(2x+1)} - \frac{x^{l+2}}{2(2x+3)} + O(x^{l+4}) \]

\[ n_{l}'(x) = \frac{x^{l+2}}{2(2x+1)} + O(x^{l+3}) \]

\[ n_{l}(x) = \frac{1}{x^{l+3/2}} - \frac{1}{x^{l+5/2}} + O\left( \frac{1}{x^{l+3}} \right) \]

\[ n_{l}'(x) = \frac{(l+1)(2x+1)}{2x+2} + O\left( \frac{1}{x^{l+2}} \right) \]

Denoting \( \chi' = k' R \), \( \chi = K R \), we have:

\[ \beta_x = \chi' \frac{j_{l}'(\chi')}{j_{l}(\chi')} = \chi' \frac{x^{l+2}}{(2x+1)} + O(x^{l+4}) \]

Numerator for \( \tan \delta_x \) is:

\[ \chi x_{l} j_{l}'(x) - \beta_x j_{l}(x) j_{l}'(x) = \frac{x^{l+2}}{(2x+1)} - \frac{(l+2)x^{l+3}}{2(2x+3)} - \frac{(l+1)x^{l+3}}{2(2x+4)} + O(x^{l+4}) \]

\[ = \frac{x^{l+2}}{(2x+3)} \left( \frac{x}{x'} - 1 \right) + O(x^{l+4}) \]
Denominator is
\[ \chi' n'(x) - \beta n(x) \chi n(x) = x \cdot \frac{(x+1)^2 x x+1}{x x+1} + A \cdot \frac{(x+1)^2 x x+1}{x x+1} + O\left(\frac{1}{x^2}\right) \]
\[ = \frac{(x+1)^2 x x+1}{x x+1} + O\left(\frac{1}{x^2}\right) \]

Thus
\[ \tan \theta = \frac{1}{(x+1)^2 x (x+3)^2} \chi (x+3) x x+3 \left(\frac{1}{x^2}\right) + O \text{ higher order terms} \]

Since
\[ \frac{y^2}{x^2} - 1 = \frac{E-V_0}{E} = \frac{V_0}{E} = \frac{2mV_0 R^2}{\hbar^2 (kR)^2} \]

We have
\[ \tan \theta = \frac{2mV_0 R^2}{\hbar^2} \frac{1}{(2n+1)(2n+3)^2} (kR)^{2n+1} + \text{higher order terms} \]

Thus
\[ \theta = \tan \theta = \sin \theta = \frac{2mV_0 R^2}{\hbar^2} \frac{1}{(2n+1)(2n+3)^2} (kR)^{2n+1} \]

\[ \delta_0 = 4\pi \frac{d\delta_0}{d\alpha} = 4\pi \frac{1}{\hbar^2} \frac{1}{\delta^2} \delta_0 = \frac{16\pi}{9} \frac{m^2 V_0^2 R^6}{\hbar^2} \]

So for small enough \( k \), the total cross section can be taken as that of S-Wave channel, and is
\[ \frac{16\pi}{9} \frac{m^2 V_0^2 R^6}{\hbar^2} \]

If we keep the result up to p-wave, then
\[ \mathcal{H}(\theta) = \frac{1}{k} \left( \delta_0 + 3 \delta_0 \cos \theta \right) \]

Then
\[ \frac{d\delta}{d\theta} = \mathcal{H}(\theta) = \frac{1}{\delta_0} \left( 1 + \frac{1}{5} (kR)^2 \cos \theta \right) \]

\[ = \frac{1}{\delta_0} \left( 1 + \frac{2}{5} (kR^2) \cos \theta \right) \]

Hence
\[ B/A = \frac{2}{5} (kR)^2 \]

Solution:

(a) In a spherically symmetric scattering potential, the partial wave expansion for the scattering amplitude \( f(\theta) \) is

\[
\begin{align*}
f(\theta) &= \frac{1}{k} \sum_{l=0}^{\infty} (2l+1) \frac{\hat{t}^l}{k^l} P_l(\cos \theta) \\
&= \sum_{l=0}^{\infty} (2l+1) \frac{\hat{t}^l}{k^l} P_l(\cos \theta), \text{ for small enough } \delta_k.
\end{align*}
\]

Use the orthogonality relation for Legendre polynomials

\[
\int_{-1}^{1} dx \ P_l(x) P_{l'}(x) = \frac{2}{2l+1} \delta_{ll'}.
\]

We get

\[
(2l+1) \frac{\hat{t}^l}{k^l} \cdot \frac{2}{2l+1} \approx \int_{0}^{\pi} \sin \theta d\theta f(\theta) P_l(\cos \theta)
\]

\[
= \int_{-1}^{1} dx \ P_l(x) \cdot (-1) \frac{2mV_0}{\hbar^2 \mu} \frac{1}{1 + \frac{\mu^2}{2\hbar^2} - x}
\]

\[
= -2mV_0 \frac{1}{\hbar^2 \mu} \int_{-1}^{1} dx \ \frac{P_l(x)}{1 + \frac{\mu^2}{2\hbar^2} - x}
\]

\[
= -2mV_0 \frac{1}{\hbar^2 \mu} Q_{\ell} \left( 1 + \frac{\mu^2}{2\hbar^2} \right)
\]

\[
\Rightarrow \delta_{\ell} = -\frac{mV_0}{\hbar^2 \mu} Q_{\ell} \left( 1 + \frac{\mu^2}{2\hbar^2} \right)
\]

(b) (ii) Obviously \( 1 + \frac{\mu^2}{2\hbar^2} > 1 \), and so the expansion of \( Q_{\ell}(x) \) applies, particularly \( Q_{\ell}(\frac{\mu}{\hbar}) > 0. \) So \( \delta_{\ell} > 0, \) \( V_0 < 0 \) and \( \delta_{\ell} < 0 \) if \( V_0 > 0. \)

(ii) When \( \frac{\mu}{\hbar} \gg \frac{\mu}{\hbar}, \) i.e., \( \frac{\mu}{\hbar} \gg 1, \) we have

\[
Q_{\ell} \left( 1 + \frac{\mu^2}{2\hbar^2} \right) \approx Q_{\ell} \left( \frac{\mu}{\hbar} \right)
\]

\[
= \frac{1}{(2\ell+1)!!} \frac{1}{\left( -\frac{\mu^2}{2\hbar^2} \right)^{\ell+1}}
\]

\[
= \frac{2^{\ell+1} \ell!}{(2\ell+1)!!} \frac{m^2}{\hbar^2 \mu^{2\ell+2}} \kappa^{2\ell+2}
\]

\[
\Rightarrow \delta_{\ell} = -\frac{2^{\ell+1} \ell!}{(2\ell+1)!!} \frac{mV_0}{\kappa^{2\ell+2} \mu^{2\ell+2}} \kappa^{2\ell+2}
\]

Solution:

(a) Radial equation for s-wave is
\[ \frac{d^2 U}{dr^2} + \left( k^2 - \frac{2m}{\hbar^2} V(r) \right) U = 0. \]

In this problem, we have
\[ \frac{d^2 U}{dr^2} + \left( k^2 - \frac{1}{r^2} \right) U = 0. \]

For \( r < R \), we have \( \frac{d^2 U}{dr^2} + k^2 U = 0. \)

Imposing the boundary condition \( U|_{r=R^0} = 0 \), one has
\[ U(r) = A \sin kr. \]

For \( r > R \), we have \( U(r) = B \sin (kr + \delta_0) \).

From the differential equation, we have
\[ \int_{R^-}^{R^+} \frac{d^2 U}{dr^2} dr = \int_{R^-}^{R^+} \left( \frac{1}{r^2} \right) U(r) dr \]
\[ = \gamma U(R) \]

Thus the connecting conditions at \( r = R \) are
\[ \begin{cases} U(r = R^-) = U(r = R^+) \\ \left. \frac{dU}{dr} \right|_{R^-} - \left. \frac{dU}{dr} \right|_{R^+} = \gamma U(R) \end{cases} \]
\[ \Rightarrow \left. \frac{U}{U} \right|_{R^+} - \left. \frac{U}{U} \right|_{R^-} = \gamma. \]

For \( r < R \), \( \left. \frac{U}{U} \right|_{R^+} = k \cot kr \); for \( r > R \), \( \left. \frac{U}{U} \right|_{R^-} = k \cot (kr + \delta_0) \).

Hence
\[ k \cot (kr + \delta_0) - k \cot kr = \gamma \]
\[ \Rightarrow \tan \delta_0 = \frac{k + \cot kr}{k + \tan kr + \cot kr} = -\frac{\frac{k}{k} \sin^2 kr}{1 + \frac{k}{k} \sin kr \cos kr} \]

So the equation for phase shift is
\[ \tan \delta_0 = -\frac{\frac{k}{k} \sin^2 kr}{1 + \frac{k}{k} \sin kr \cos kr} \]

(b) 1) (If \( \tan kr \) is not close to zero, one gets hard sphere scattering.)

Using \( \gamma \gg k \), i.e. \( \frac{k}{k} \gg 1 \), we get
\[ \tan \delta_0 \approx \frac{\frac{k}{k} \sin^2 kr}{1 + \frac{k}{k} \sin kr \cos kr} \]

which is just the hard sphere result.

2) (Resonance)

Resonance occurs when cross section for the partial wave channel reaches its maximal value
while at the same time \( \cot \theta = -\frac{1}{\frac{2k}{\sin kr}} \), goes through zero from positive side as \( k \) increases.

\[
\cot \theta = \frac{1 + \frac{k}{\sin kr} \cos \theta}{\frac{k}{\sin kr}} = \frac{1 + \frac{k}{\sin kr}}{\frac{k}{\sin kr} \cos \theta} = \frac{\frac{k}{\sin kr}}{\frac{k}{\sin kr} \cos \theta} = -\frac{1}{2} \frac{\sin 2kr + \frac{2k}{\sin kr}}{\sin 2kr \sin \theta} = -\frac{1}{2} \frac{\sin 2kr - (-\frac{2k}{\sin kr})}{\sin 2kr \sin \theta}
\]

Let \( \cot \theta = 0 \), we have \( \sin 2kr = -\frac{2k}{\sin kr} \to 0 \), \( r \to \infty \). So for large \( r \), \( \sin 2kr \) is very close to \( 0 \). If we require \( \cot \theta \) to pass through zero from a positive side then \( \sin 2kr - (-\frac{2k}{\sin kr}) \) has to pass through negative side.

![Graph of \( \sin 2kr \) vs \( kr \)]

The slope \( -\frac{2k}{\sin kr} \ll 2k \) since \( r \gg \frac{1}{Y} \). From graph we see that \( B \) is true solution while \( A \) is not, so \( 2kr = 2n\pi + \chi \), where \( \chi \) is small.

Then \( 2kr = \frac{2k}{1/2} \), or \( kr = n\pi - \frac{k}{Y} \).

Hence \( \sin (2kr) = \sin x = -\frac{2k}{Y} \Rightarrow x = \frac{2k}{Y} \).

3. (Determine position of resonance to order \( \frac{1}{Y} \); compare result with spherical well)

Resonance position has been determined in 3 as \( kr = n\pi - \frac{k}{Y} \),
or \( kr = \frac{n\pi}{1 + \frac{k}{Y}} \Rightarrow n\pi (1 - \frac{1}{Y}) \).

For the quantum well, let \( \sin kr = n\pi \), we get \( kr = n\pi \).

One can see that positions of resonance is quite close to bound states in a quantum well in the limit of large \( r \).

4. (Obtain an expression for resonance width)

\[
\frac{d\theta}{dE} = \frac{d}{dE} \left( \frac{1 + \frac{k}{\sin kr} \cos \theta}{\frac{k}{\sin kr}} \right)
\]

\[
= -\frac{1}{k \sqrt{Y}} \frac{d}{dE} \left( \frac{1 + \frac{k}{\sin kr} \cos \theta}{\frac{k}{\sin kr}} \right)
\]

\[
= -\frac{m}{k \sqrt{Y}} \frac{d}{dE} \left( \frac{k + \frac{k}{\sin kr} \cos \theta}{\frac{k}{\sin kr}} \right)
\]

\[
= -\frac{m}{k \sqrt{Y}} \frac{1}{\sin kr} \left[ Y \sin kr (1 + \frac{Yr}{\sin kr} \cos \theta - \sin kr) \right] - \frac{m}{k \sqrt{Yr}} \left( k + \frac{k}{\sin kr} \cos \theta - \sin kr \right)
\]
\[- \frac{m}{h^2 k} \cdot \frac{1}{Y \sin kR} \left[ \sin kR \left( 1 + \frac{Y R (\cos kR - \sin kR)}{\sin kR} \right) \right] - 2k \cos kR \left( k + Y \sin kR \cos kR \right) \]

At \( E = E_r \), i.e. \( k = k_r = \frac{m a}{RY} \), we can replace \( \sin kR \) with \( \frac{\pi R k}{Y} \), \( \cos kR \) with \( (-1)^n \).

Then

\[
\frac{d \psi}{dE} \bigg|_{E = E_r} = \frac{m}{\hbar^2 k} \cdot \frac{1}{Y} \left( \frac{\pi R k}{Y} \right)^3 \cdot (-1)^n \cdot 2kR \cdot (-1)^n.
\]

\[
\Rightarrow \quad \Gamma = \frac{\hbar^2 Y (\pi a)^3}{m R} = \frac{\hbar^2 \pi^3}{m R^2 Y^2} \propto \frac{1}{Y^2}
\]

So \( \Gamma \) decreases as \( Y \) increases.