p-wave Cooper pairing and more

The most celebrated example of the p-wave Cooper pairing is the $^3$He. Except that it's charge neutral and thus the EM response is different, they are very similar to paired superconductors. The solid-state p-wave system is $Sr_2RuO_4$, and ultra-cold dipolar fermions also gives rise to p-wave pairing. P-wave pairing has an enormously rich structure, $L=1$, $S=1$.

1. isotropic - B phase $J = L + S = 0$.
   - fully gapped, 3D topological pairing

2. anisotropic - A phase $J$ is not well-defined.
   - nodal quasi-particle

3. $J$-triplet pairing (Y. Li and C. Wu)
   - a new pairing pattern $J = L = S = 1$ due to dipolar interaction
   - $J_2 = 0$
   - $\Delta(k) \propto \sin \theta_k$

$J_2 = \pm 1$

- $\Delta(k) \propto 1 \pm \cos \theta_k$

We use the continuum model

\[ H = \sum_k (\varepsilon(k) - \mu) a_{k\alpha}^+ a_{k\alpha} + \frac{1}{2Vd} \sum_{k,k'} V(k,k') \sum_{\alpha,\beta} a_{k\alpha}^+ a_{k'\beta} a_{-k'\alpha} a_{-k\beta} \]

and we use a factorizable interaction: \( V(k,k') = -V \vec{k} \cdot \vec{k}' \).

(This pairing interaction mainly arise from ferro-magnetic fluctuati)

\[
\Delta^{a\alpha_0}_0 = -\sum_{k'} V(k,k') < a_{k'\alpha} a_{-k'\alpha} > \\
= \Delta^{a\mu}_0 \cdot (\sigma_\mu i \sigma_2)_{0\alpha_0} \\
\mu - \text{spin channel} \quad \alpha - \text{orbital channel}
\]

Thus the p-wave order parameter \( 3 \times 3 \) amplex matrix, which has 18 real parameters.

We can also define the pairing matrix \( \Delta_{0\alpha_0}(k) = k a \Delta^{a\alpha_0}_0 \).

\[
\Delta_{0\alpha_0}(k) = \Delta^{a\mu}_0 k a \cdot (\sigma_\mu i \sigma_2)_{0\alpha_0} = \Delta(k) \hat{d}(k) (\sigma_\mu i \sigma_2)_{0\alpha_0} \\
\text{the tensor } \Delta^{a\mu}_0 \text{ maps the momentum } \vec{k} \text{ into a vector in spin channel} - d \text{-vector}.
\]

\( \Delta(k) \) is a complex number, the spin structure of Cooper pair is discribed by the d-vector.

\[
\hat{d}(k) \tilde{d}(k) = \sum_{\mu} \hat{d}_{\mu}(k) \hat{d}_{\mu}(k) = 1
\]

The \( \hat{d}(k) \) vector is normalized as
using d-vector, \( \Delta_{oo'}(k) = \Delta(k) \begin{pmatrix} -\hat{d}_x(k) + i\hat{d}_y(k) \\ \hat{d}_z(k) \end{pmatrix} \begin{pmatrix} \hat{d}_x(k) + i\hat{d}_y(k) \\ \hat{d}_z(k) \end{pmatrix} \) \(^3\)

**\( \Delta_{oo'}(k) \)** is a symmetric matrix, (triplet)

- in comparison, the singlet channel pairing \( \Delta_{oo'} = \Delta_s (\hat{z}_0)_{oo'} = \begin{pmatrix} 0 & \Delta_s \\ -\Delta_s & 0 \end{pmatrix} \)

is anti-symmetric.

- physical meaning of d-vector

  In many situations, \( \hat{d}(k) \) up to an overall phase can be chosen as real, and we attribute the phase to \( \Delta(k) \). Nevertheless, the direction of \( \hat{d}(k) \) is not well-defined: if we set \( \hat{d}(k) \to -\hat{d}(k) \) then \( \Delta(k) \) and \( \Delta(k) \) are invariant!

  Thus \( \hat{d} \)-vector is actually a director, not a real vector.

  The physical meaning of d-vector: if \( \hat{d}(k) \) is real, then \( \hat{d}(k) \) is not the spin direction of the Cooper pair. For example, if \( \hat{d}(k) = \hat{z} \), it means the pairing \( \Delta_{oo'} = \Delta_s \langle \hat{a}_k \hat{a}_{-k} + \hat{a}_k \hat{a}_{-k} \rangle \) which's in the total spin \( S=1, S_z=0 \). The spin actually fluctuates in the \( x-y \) plane. Thus \( \hat{d}(k) \) is perpendicular to the spin, or, \( \hat{d}(k) \) is the direction such that \( \hat{d} \cdot \hat{\Sigma} \) is in the eigenstate with \( \hat{d} \cdot \hat{\Sigma} = 0 \). For such a state all the spin average value is zero.
However, if \( \hat{\delta} \) is complex, or, \( \text{Re} \hat{\delta} \neq \text{Im} \hat{\delta} \), then the angular momentum expectation value of Cooper pair is nonzero. Let us consider pairing \( a_k^+ a_{-k}^+ \), which corresponding to \( \hat{\delta} = \frac{1}{\sqrt{2}} (1, i, 0) \), then \( S_z = 1 \).

\[
\hat{\delta}^* \times \hat{\delta} = i \Rightarrow \vec{\mathbf{s}} = -i \hat{\delta}^* \times \hat{\delta}
\]

Ex: prove

\[
\langle \hat{S}_z(k) \rangle = -i \hat{\delta}^* \times \hat{\delta} \left| \Delta(k) \right|^2
\]

for a triplet Cooper pair described by \( \Delta_{\sigma \sigma'}(k) = \Delta(k) \hat{\delta}_a^{(\sigma \sigma')} \delta_{\sigma \sigma'}/\sqrt{2} \).

\* Bogoliubov spectra (mean-field Hamiltonian)

\[
\hat{H}_{MF} = \sum_{k_0} (\epsilon_k - \mu) a_{k_0}^+ a_{k_0} - \frac{1}{2} \sum_{k_0} a_{-k_0}^+ a_{-k_0}^+ a_{-k_0} a_{k_0} - k a \Delta_{\sigma \sigma'}^a
\]

\[
- \frac{1}{2} \sum_{k, k_0} a_{-k_0}^+ (\Delta_{\sigma \sigma'}^a a_k) \delta_{\sigma \sigma'} a_{k_0}
\]

\[
+ \frac{\text{Vol}}{2 V_0} \sum_{\sigma \sigma', a} | \Delta_{\sigma \sigma'}^a |^2
\]
using the property $\Delta_{oo'}(-k) = -\Delta_{oo'}(k)$ (please check),
we can simplify

$$\frac{1}{2} \sum_{k_{oo'}} \left[ a^+_{k_0} \Delta_{oo'}(k) a^+_{-k_0} \right] = \sum_{k_{oo'}} \left[ a^+_{k_0} \left( \Delta_{oo',ka}^a a^+_{-k_0} \right) \right].$$

$\Sigma'$ means only sum over half of the momentum space

$$\Rightarrow H_{MF} = \sum_{k_{oo'}} \left( a^+_{a'} a^+_{a} a a' \right) H_{\omega\beta}(k) \begin{pmatrix} a_{a'} \\ a_{a} \\ a^+_{a'} \\ a^+_{a} \end{pmatrix} + \frac{Vol}{2V_e} \sum_{oo',a} \left| \Delta_{oo'}^a \right|^2$$

$$H_{\omega\beta}(k) = \begin{bmatrix} E(k) - \mu & \Delta_{oo'}(k) \\ \Delta^+(k) & -(E(k) - \mu) \end{bmatrix}, \text{ where } \Delta_{oo'}(k) = \Delta(k)$$

For simplicity, we set $\Delta(k)$ and $\hat{a}$ real, $H_{\omega\beta}(k)$ can be expressed in terms of $\Gamma$-matrix

$$H_{\omega\beta}(k) = (E(k) - \mu) \Gamma^1 + \Delta(k) \left[ d^+_x(k) \Gamma^3 + d^+_y(k) \Gamma^4 + d^+_z(k) \Gamma^5 \right]$$

$$\Gamma^1 = 1 \otimes \tau_3, \Gamma^2 = \sigma_2 \otimes \tau_1, \Gamma^3 = \sigma_3 \otimes \tau_1, \Gamma^4 = 1 \otimes \tau_2, \Gamma^5 = -\sigma_1 \otimes \tau_1$$

$\tau$ refers to the particle–hole channel

$\sigma$ refers to spin

$$H^2(k) = (E(k) - \mu)^2 + \Delta^2(k) \Rightarrow E(k) = \pm \sqrt{(E(k) - \mu)^2 + \Delta^2(k)}$$
For the B-phase, the d-vector: \( \Delta \mathbf{v}(k) = \Delta(k) \hat{d}_\mu(k) \left( \hat{\sigma}_\mu \hat{\sigma}_0 \right) \).

Thus \( \Delta \mu a \) maps the momentum space vector \( \hat{k} \) to a vector in spin space. If \( \Delta \mu a \) proportional to a \( O(3) \) matrix, i.e., \( \Delta \mu a \propto \delta_{\mu a} \), then realizes a connection between two triads. In the simplest case \( \delta_{\mu a} \propto \delta_{\mu a} \).

\[ \hat{d}(k) = \hat{k} \]

\(^3\text{He-B} \) is an isotropic phase, i.e.

\[ J = L + S = 0. \]

We need to co-rotate spin and momentum together, i.e. spin-orbit coupling (p-p channel)

Spontaneously breaking of spin-orbit symmetry.

Goldstone mode / manifold \( SO(3) \times SO(3) / SO(3) \)

relative spin-orbit rotation, i.e. the degree of freedom \( \delta_{\mu a} \),

\[ \sum_{\mu a} \delta_{\mu a} \cdot \delta_{\mu a} = 1. \]

The spectra is fully gapped: \[ E(k) = \pm \sqrt{(\varepsilon(k) - \mu)^2 + |\Delta|^2}. \]
2) The A-phase: \[ \Delta_{\alpha'}(k) = \Delta(k) \hat{d}_\mu(k) (\hat{\sigma} \cdot \hat{v}_z) \delta_{\alpha} \]

\[ \Delta(k) \hat{d}_\mu(k) = \Delta e^{i\theta} \hat{d}_\mu \{ (\hat{e}_1 + i\hat{e}_2) \cdot \hat{k} \} \]

\( \hat{d} \)-vector is momentum-independent, but \( \Delta(k) \) depends on \( \hat{k} \), \( (\hat{p}_x + i\hat{p}_y) \) rotate \( \vec{I} = \hat{e}_1 \times \hat{e}_2 \)

direction of orbital angular momentum.

Rotation of the frame \( \hat{e}_1, \hat{e}_2 \) around \( \vec{I} \)-vector at angle \( \alpha \), is equivalent to a phase gauge transformation.

\[ \hat{e}_1 + i\hat{e}_2' = e^{i\alpha} (\hat{e}_1 + i\hat{e}_2) \]

\[ \Rightarrow \Delta'(k) = \Delta(k) e^{i\alpha} \]

Now let us set \( \hat{e}_1 = \hat{x}, \hat{e}_2 = \hat{y}, \vec{I} = \hat{z}, \hat{d}_\mu = \hat{z} \Rightarrow \]

\[ |\Delta(k)|^2 = |1I|^2 (\hat{k}_x^2 + \hat{k}_y^2) = |1I|^2 \sin^2 \Theta_k \]

\[ \Rightarrow E(k) = \pm \sqrt{(E(k) - \mu)^2 + |1I|^2 \sin^2 \Theta_k} \]

Dirac fermion at \( \Theta = 0, \) and \( \pi \).
Green's function (Matsubara)

\[
\begin{bmatrix}
- T_\varepsilon < \alpha_\varphi (k_0^2) \varphi_\alpha (k, 0) >,
- T_\varepsilon < \alpha_\varphi (k_0^2) \alpha_\varphi (-k, 0) > \\
- T_\varepsilon < \alpha_\varphi (-k_0^2) \varphi_\alpha (k, 0) >,
- T_\varepsilon < \alpha_\varphi (-k_0^2) \alpha_\varphi (-k, 0) >
\end{bmatrix}
\]

it's Fourier transform \( \Rightarrow [i\omega_n - H_{0\beta}(k)]^{-1} = G(k, i\omega_n) \)

\[
G(k, i\omega_n) = \begin{bmatrix}
G_{\varphi\varphi}(k, i\omega_n) & F_{\varphi\alpha}(k, i\omega_n) \\
F_{\alpha\varphi}^+(k, i\omega_n) & -G_{\alpha\alpha}(-k, -i\omega_n)
\end{bmatrix}
\]

\[
= \frac{i\omega_n + (E(k) - \mu) \Gamma^1 + \Delta(k)(dx^1 P_3 + dy^1 P_4 + dz^1 P_5)}{(i\omega_n)^2 - E(k)}
\]
Solution for edge modes ($P + \frac{i}{2} P / \text{He-3B}$).

1. Simplified model

\[
\begin{bmatrix}
-\mu(x) & \frac{\Delta(-i\partial_x + ik_y)}{k_f} \\
\frac{\Delta(-i\partial_x - ik_y)}{k_f} & \mu(x)
\end{bmatrix}
\begin{bmatrix}
U_n \\
V_n
\end{bmatrix}
\begin{bmatrix}
e^{ik_y y} \\
e^{ik_y y}
\end{bmatrix}
= \begin{bmatrix}
U_n \\
V_n
\end{bmatrix}
\begin{bmatrix}
e^{ik_y y} \\
e^{ik_y y}
\end{bmatrix}
\]

\[
\begin{align*}
\begin{cases}
-\mu(x) U_n + \frac{\Delta(-i\partial_x V_n + ik_y V_n)}{k_f} = E_n(k_y) U_n, & \mu(x) > 0 \\
\frac{\Delta(-i\partial_x U_n - ik_y U_n)}{k_f} + \mu(x) V_n = E_n(k_y) V_n, & \mu(x) < 0
\end{cases}
\end{align*}
\]

We are only interested in the edge states. These states are zero mode along the $x$-direction. The dispersion purely comes from the plane-wave along $y$-direction. We should try

\[
\begin{align*}
\begin{cases}
\frac{\Delta}{k_f} i k_y U_0 = E_0(k_y) U_0, \\
\frac{\Delta}{k_f} (i k_y U_0) = E_0(k_y) U_0
\end{cases}
\Rightarrow
\begin{cases}
U_0 = -i U_0 \\
E_0(k_y) = -\Delta k_y / k_f
\end{cases}
\end{align*}
\]

but actually only one is possible.

We need to check the zero mode along the $x$-direction should be localized at $x = 0$.

Set $U_0 = -i U_0 \Rightarrow (-\mu(x) + \frac{\Delta}{k_f} \partial_x) U_n = 0$ from 1st Eq

\[
\begin{align*}
\begin{cases}
\frac{\Delta}{k_f} \partial_x U_n - \mu(x) U_n = 0
\end{cases}
\end{align*}
\]

\[
\Rightarrow \text{these two Eqs are consistent.}
\]
\[ \frac{1}{k_f} \partial_x U_0 = \frac{\mu(x)}{\Delta} U_0 \quad \Rightarrow \quad U_0(x) = e^{-\int_0^x \frac{dx'}{k_f} |\mu(x')|} \]

For the current set up, that \( \mu(x) < 0 \) at \( x > 0 \), we do have exponential decay solution. The other try that \( U_0 = iU_0 \) does not work, which gives rise to exponentially divergent solutions.

1) Now let us restore the dispersion \( H_0 = f_y(k_y) + f_x(-i\hbar \partial_x) - \mu(x) \) we have
\[ \frac{\Delta}{k_f} (-i \partial_x U_0 - i\hbar \partial_y U_0) + [-f_y(k_y) - f_x(-i\hbar \partial_x) + \mu(x)] U_0 = E_0(k_y) U_0 \]

Still try the solution
\[ \begin{cases} \frac{\Delta}{k_f} i \hbar \partial_y U_0 = E_0(k_y) U_0 \\ \frac{\Delta}{k_f} (-i) \partial_y U_0 = E_0(k_y) U_0 \end{cases} \] (let's choose \( U_0 = -iU_0 \))

and the \( x \)-direction
\[ \left[ f_y(k_y) + f_x(-i\hbar \partial_x) - \mu(x) \right] U_0 + \frac{\Delta}{k_f} \partial_x U_0 = 0 \quad \text{from 1} \]
\[ \left[ \frac{\Delta}{k_f} \partial_x + f_y(k_y) + f_x(-i\hbar \partial_x) - \mu(x) \right] U_0 = 0 \quad \text{from 2} \]

consistent! \( \Rightarrow \) the edge spectra is not affected, which \( E_0(k_y) \) is still determined by the off-diagonal term \( E(k_y) = \frac{-\Delta k_y}{k_f} \).
but the zero mode Eq along the x-direction →

\[ \left[ \frac{\partial}{\partial x} (ix \partial_x) + \frac{\Delta}{\hbar_f} \partial_x \right] U_0 = \left[ \mu(x) - f_y(k_y) \right] U_0 \]

or

\[ \left[ -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + \frac{\Delta}{\hbar_f} \partial_x \right] U_0 = \left[ \mu(x) - \frac{\hbar^2 k_y^2}{2m} \right] U_0 \]

This Eq is more realistic compared with the oversimplified one

\[ \frac{\Delta}{\hbar_f} \partial_x U_0 = \mu(x) U_0. \]

In that case, all the states \((k_x, k_y)\) in the bulk plane wave are occupied, i.e. \(\hbar_f \to +\infty\). Now, if for the value of \(\hbar_y\), such that \(\frac{\hbar^2 k_y^2}{2m} > \mu\) (see figure), we have no edge states, (because \(\mu(x) - \frac{\hbar^2 k_y^2}{2m}\) always negative).

\[ \begin{array}{c} \mu(k_x) \\ \mu_L > 0 \end{array} \]

\[ \begin{array}{c} \mu_R < 0 \end{array} \]

\[ x \]

\[ \text{system} \] \[ \text{vacuum} \]

\[ E(k_y) = \frac{\hbar^2 k_y^2}{2m} + \frac{\hbar^2 k_x^2}{2m} \]

\[ \Delta = 0 \]

Each parabola is with a different \(k_x\). (say, \(k_x = \frac{n\pi}{L_x}\) for open boundary).

\[ \Delta \neq 0 \]

\[ \text{edge states} \]

\[ \frac{U}{U_f} = \frac{\Delta}{\hbar_f v_f} \approx \frac{\Delta}{E_f} \]

\[ \text{estimation of edge state velocity} \]
Surface states of the BW state

\[ H = \left[ \begin{array}{cc} -\frac{\hbar^2}{2m} \nabla^2 + \mu(x) & \Delta (-i\hbar \nabla \cdot \hat{\sigma}) i \sigma_z \end{array} \right] \left[ \begin{array}{c} \phi_1 \\ \phi_2 \end{array} \right] = E \left[ \begin{array}{c} \phi_1 \\ \phi_2 \end{array} \right] \]

Seek

\[ \begin{array}{c} \phi_1(z) \\ \phi_2(z) \end{array} = e^{ik_x x + i k_y y} \]

\[ \left[ \begin{array}{c} -\frac{\hbar^2}{2m} \nabla^2 - \mu(x) & \Delta \left( -i \hbar \nabla \cdot \hat{\sigma} \right) \left( \hbar k_x \sigma_1 + \hbar k_y \sigma_2 - i \hbar \partial_2 \sigma_3 \right) i \sigma_z \end{array} \right] \left[ \begin{array}{c} \phi_1 \\ \phi_2 \end{array} \right] = E_{0} (k_x, k_y) \left[ \begin{array}{c} \phi_1 \\ \phi_2 \end{array} \right] \]

Surface state spectra:

\[ \left\{ \begin{array}{c} \Delta \left( \hbar k_x \sigma_1 + \hbar k_y \sigma_2 - i \hbar \partial_2 \sigma_3 \right) \phi_1 + \left( -\frac{\hbar^2}{2m} \nabla^2 + \frac{\hbar^2}{2m} \frac{\partial^2}{\partial z^2} + \mu(x) \right) \phi_2 = E_{0} (k_x, k_y) \phi_2 \\
\Delta (-i \sigma_z) \left( \hbar k_x \sigma_1 + \hbar k_y \sigma_2 - i \hbar \partial_2 \sigma_3 \right) \phi_1 + \left( -\frac{\hbar^2}{2m} \nabla^2 + \frac{\hbar^2}{2m} \frac{\partial^2}{\partial z^2} + \mu(x) \right) \phi_2 = E_{0} (k_x, k_y) \phi_2 \end{array} \right\} \]

we want

\[ \Delta \hbar \left( \hbar k_x \sigma_1 + \hbar k_y \sigma_2 \right) i \sigma_z \phi_2 = E_{0} (k_x, k_y) \phi_2 \quad 0 \]

\[ \Delta (-i \sigma_z) \hbar \left( \hbar k_x \sigma_1 + \hbar k_y \sigma_2 \right) \phi_1 = E_{0} (k_x, k_y) \phi_2 \quad 0 \]

\[ \phi_1 = T \phi_2 \Rightarrow \Delta \hbar \left( \hbar k_x \sigma_1 + \hbar k_y \sigma_2 \right) i \sigma_z \phi_2 = E_{0} (k_x, k_y) T \phi_2 \]

or

\[ \Delta \hbar \left( \hbar k_x \sigma_1 + \hbar k_y \sigma_2 \right) i \sigma_z \phi_2 = E_{0} (k_x, k_y) \phi_2 \]

\[ \Rightarrow \Delta \hbar \left( -i \sigma_z \right) \left( \hbar k_x \sigma_1 + \hbar k_y \sigma_2 \right) T \phi_2 = E_{0} (k_x, k_y) \phi_2 \]
\[
T^{-1}(k_x \sigma_1 + k_y \sigma_2) \ i \sigma_2 = (-i \sigma_2)(k_x \sigma_1 + k_y \sigma_2) T
\]

Also need to be Hermitian.

\[
T^{-1} i \sigma_2 (-k_x \sigma_1 + k_y \sigma_2) = (-i \sigma_2) \ T^{-1} (k_x \sigma_1 + k_y \sigma_2) T
\]

We need \(-k_x \sigma_1 + k_y \sigma_2 \propto T^{-1} (k_x \sigma_1 + k_y \sigma_2) T\)

We can set \(T \propto \) either \(\sigma_1\), or \(\sigma_2\), but not \(\sigma_3\).

If we set \(T \propto \sigma_2\), we have

\[
T^{-1} (k_x \sigma_1 + k_y \sigma_2) T = (-k_x \sigma_1 + k_y \sigma_2)
\]

\[
\Rightarrow T^{-1} i \sigma_2 = (-i \sigma_2) T \Rightarrow T = i \sigma_2
\]

If \(T = i \sigma_2\), i.e. \(\phi_1 = i \sigma_2 \phi_2\)

\[
\begin{bmatrix}
\frac{\hbar^2 k_x^2}{2m} - \frac{\hbar^2 \partial^2}{2m \partial z^2} - \mu(x)
\end{bmatrix}
\]

\[
\begin{bmatrix}
\Delta (i \hbar \partial_2 \sigma_2) i \sigma_2 \phi_2 = 0
\end{bmatrix}
\]

\[
\begin{bmatrix}
\Delta (-i \sigma_2) (-i \hbar \partial_2 \sigma_2) i \sigma_2 \phi_2 + \left[ -\frac{\hbar^2 k_x^2}{2m} + \frac{\hbar^2 \partial^2}{2m \partial z^2} + \mu(x) \right] \phi_2 = 0
\end{bmatrix}
\]

This means that \(\phi_2\) has to satisfy another matrix Eq. This is not consistent with

\[
(-i \sigma_2)(k_x \sigma_1 + k_y \sigma_2)(i \sigma_2) \phi_2 = E(k_x, k_y) \phi_2
\]

\[
(-k_x \sigma_1 + k_y \sigma_2) \phi_2 = E(k_x, k_y) \phi_2
\]
In other words, we seek a purely scalar equation for the \( z \)-direction.

The choice of \( \phi_1 = i \sigma_z \phi_2 \) doesn't work!

Instead, we choose \( \phi_1 = \pm i \omega \phi_2 \) (\( \pm \) signs apply to different boundary).

\[
\left[ \frac{-h^2 k_{11}^2}{2m} - \frac{h^2}{2m} \frac{\partial^2}{\partial z^2} - \mu(x) \right] \phi_1 - \Delta \left( -i \hbar \partial_z \sigma_3 \right) \left( i \omega \right) \left( \pm i \sigma_i \right) \phi_1 = 0
\]

\[
\Delta \left( -i \omega \right) \left( -i \hbar \partial_z \sigma_3 \right) \phi_1 + \left[ \frac{-h^2 k_{11}^2}{2m} + \frac{h^2}{2m} \frac{\partial^2}{\partial z^2} + \mu(x) \right] \left( \mp i \sigma_i \right) \phi_1 = 0
\]

\[
\Rightarrow \left[ \frac{-h^2 k_{11}^2}{2m} - \frac{h^2}{2m} \frac{\partial^2}{\partial z^2} - \mu(x) \right] \phi_1 + \Delta \left( \mp i \sigma_i \right) \left( -i \omega \right) \left( -i \hbar \partial_z \sigma_3 \right) \phi_1 = \mp \Delta \hbar \partial_z \phi_1
\]

\[
\Rightarrow \text{consistent}
\]

\[
\left[ \frac{-h^2 k_{11}^2}{2m} - \frac{h^2}{2m} \frac{\partial^2}{\partial z^2} - \mu(x) \right] \phi_1 = \mp \Delta \hbar \partial_z \phi_1 = 0
\]

\[
\left[ -\frac{h^2}{2m} \frac{\partial^2}{\partial z^2} \mp \frac{\Delta}{\hbar} \partial_z \right] \phi_1 = \left[ \mu(x) - \frac{h^2 k_{11}^2}{2m} \right] \phi_1
\]

which is the same as before

\[
\phi_1 = \pm i \omega \phi_2
\]

\[
\mp \Delta \left( k_x \sigma_1 + k_y \sigma_2 \right) i \omega \left( \pm i \sigma_i \right) \phi_1 = E_0(k_x, k_y) \phi_1
\]

\[
\mp \Delta \left( k_x \sigma_2 - k_y \sigma_1 \right) \phi_1 = E_0(k_x, k_y) \phi_1
\]
Now let us solve the normal direction: we use the 2D case.

\[
\begin{align*}
[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{\Delta}{k_F} \frac{\partial}{\partial x}] U_o &= [\mu_L - \frac{\hbar^2 k^2}{2m}] U_o \quad \text{for} \ x < 0, \ \text{where} \ \mu_L = \frac{\hbar^2 k^2}{2m}, \\
[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{\Delta}{k_F} \frac{\partial}{\partial x}] U_o &= [\mu_R - \frac{\hbar^2 k^2}{2m}] U_o \quad \text{for} \ x > 0
\end{align*}
\]

Actually, if both \(\mu_L\) and \(\mu_R > 0\), but \(\mu_L > \mu_R\),

there may still exist \(\mu_L > \frac{\hbar^2 k^2}{2m} > \mu_R\), such

that are edge states. This may also be interesting and need check further!

Generally, speaking since 2nd order derivatives are involved, and \(\mu_L\) and \(\mu_R\) steps are finite, we expect non-continuity of \(U_0''(x)\),

but \(U'(x)\), and \(U(x)\) are continuous at the boundary. Imagine that we set \(\mu_R \to -\infty\), which corresponds to open boundary, i.e. \(U_0(x) = 0\) for \(x > 0\).

Then \(U_0(x)\) may also be discontinuous, \(U_0(x = 0^+) - U_0(x = 0^-)\)

but \(U_0(x)\) should be continuous,

\[
\int_0^{0^+} dx \ U'' \to c \ U'' \quad \text{finite may be}
\]

i.e. we seek solution

\[
U_0(0) = 0, \ \text{and} \ U_0(-\infty) = 0.
\]

Let us try \(U_0 \sim e^{\beta x}\) for \(x < 0\), where \(\text{Re} \ \beta > 0\). (we consider

the left space, so \(\text{Re} \ \beta > 0\)).

\(\beta\) can actually be complex.
\[ \frac{-\Delta p^2}{2m} + \frac{\Delta}{k_f} p = \mu - \frac{\Delta k_f^2}{2m} \Rightarrow \left( \frac{\beta}{k_f} \right)^2 - \frac{\Delta (\beta)}{E_F (\frac{\beta}{k_f})} + \left( 1 - (\frac{k_f}{k_f})^2 \right) = 0 \]

for the usual case that \( \frac{\Delta}{E_F} \ll 1 \).

If \( \frac{k_f}{k_f} \ll 1 \), we have \( \left( \frac{\Delta}{E_F} \right)^2 - 4 \left[ 1 - (\frac{k_f}{k_f})^2 \right] < 0 \)

or for \( \left| \frac{k_f}{k_f} \right| < \sqrt{1 - \left( \frac{\Delta}{E_F} \right)^2} \), the solutions \( \beta \) is a pair of complex variables. \( \Rightarrow \)

\[ \frac{\beta}{k_f} = \frac{1}{2} \frac{\Delta}{E_F} \pm i \sqrt{\left[ 1 - \left( \frac{k_f}{k_f} \right)^2 \right] \left( \frac{\Delta}{E_F} \right)^2} \]

We seek

\[ u_0(x) \sim e^{\frac{k_f \Delta}{2 E_F} x} \cdot \sin \left( \sqrt{\left[ 1 - \left( \frac{k_f}{k_f} \right)^2 \right] \left( \frac{\Delta}{E_F} \right)^2} \cdot k_f x \right) \]

in the case of \( \frac{k_f \Delta}{2 E_F} \gg \sqrt{\left[ 1 - \left( \frac{k_f}{k_f} \right)^2 \right] \left( \frac{\Delta}{E_F} \right)^2} \), then the oscillation is cut off by the exponential decay, we can approximate \( \sin \# x \sim \# x \)

\[ u_0(x) \sim x e^{\frac{k_f \Delta}{2 E_F} x} \quad \text{up to an overall normalization} \]

(\( \beta_1, \beta_2 = \frac{1}{2} \frac{\Delta}{E_F} \pm \sqrt{\left( \frac{\Delta}{E_F} \right)^2 - \left[ 1 - (\frac{k_f}{k_f})^2 \right]} \))

\( \\text{If } \left( \frac{\Delta}{E_F} \right)^2 - 4 \left[ 1 - (\frac{k_f}{k_f})^2 \right] > 0 \) and \( \left| \frac{k_f}{k_f} \right| \leq 1 \), we have 2 real roots positive

\( \Rightarrow \left| \frac{k_f}{k_f} \right| \geq \sqrt{1 - \left( \frac{\Delta}{E_F} \right)^2} \)
We seek \( u_0(x) = e^{\beta_1 x} - e^{\beta_2 x} = e^{\beta x} \),
\( \Theta \left( e^{\frac{(\beta_1 - \beta_2)x}{2}} - e^{\frac{(\beta_1 - \beta_2)x}{2}} \right) \)

a) as \( \left| \frac{k_y}{k_F} \right| \sim \sqrt{1 - \left( \frac{\Delta}{2E_F} \right)^2} \), \( |\beta_1 - \beta_2| \ll \beta_2 \).

again in this case, the decay is dominated by \( e^{\beta_2 x} \), and
\[ e^{(\beta_1 - \beta_2)x} \sim \frac{\beta_1 - \beta_2}{2} \sim (\beta_1 - \beta_2) x, \Rightarrow u_0(x) \sim x e^{\frac{k_F}{2E_F} x} \]

b) as \( \left| \frac{k_y}{k_F} \right| \to 1 \), \( \beta_2 \ll \beta_1 \), thus the decay becomes slow
\[ u_0(x) = e^{\beta_1 x} - e^{\beta_2 x} = e^{\beta_2 x} \left( 1 - e^{-(\beta_1 - \beta_2)x} \right) \]
\[ \begin{cases} 
\alpha x e^{\beta_2 x} \\
\alpha e^{\beta_2 x} \quad \text{decay length } \frac{1}{\beta_2} \to \infty
\end{cases} \]
and merge to bulk states.

\[ \sqrt{(\Delta)^2 - (1 - \left| \frac{k_y}{k_F} \right|^2)} = \left[ \left( \frac{\Delta}{2E_F} \right)^2 - 2 \left( 1 - \frac{k_y}{k_F} \right) \right]^{1/2} = \frac{\Delta}{2E_F} - \frac{\Delta - \frac{k_y}{k_F}}{2E_F} \]

\( \beta_2 \sim \frac{k_y - 1}{k_F} \frac{k_y}{k_F} \).

\( \text{if } k_y > k_F, \text{ two real roots. One positive, one negative. } \)

\( \text{no way to form a solution } u_0(0) = u_0(-\infty) = 0. \text{ No edge states. } \)
If $\Delta$ is so large (unrealistic), such that $\frac{\Delta}{E_f} \geq \infty$, then we for the entire $|\Delta| > |k_g/k_f| > 0$, we have always

$$\rho_{1,2}/k_f = \frac{1}{\alpha} \frac{\Delta^2}{E_f} \pm \sqrt{(\frac{\Delta}{E_f})^2 - 1 + (\frac{k_g}{k_f})^2}.$$  

The decay length is determined by $\sqrt{\beta_k k_f}$. 