lecture 6 path integral to superfluid.

For a free boson system $H_0 = \sum_k (E_k - \mu) a_k^\dagger a_k \to$

$$\int D[a_k(x)] \ e^{i \int dt \ \sum_k \left[ i \frac{1}{2} (a_k^\dagger \dot{a}_k - \dot{a}_k a_k^\dagger) - (E_k - \mu) a_k^\dagger a_k \right]}$$

in real-space, we have

$$\int D[a] \ e^{i \int d^d x dt \ \left[ \frac{1}{2} (\phi^*(x) \partial_t \phi(x) - \phi(x) \partial_t \phi^*(x)) - \frac{1}{2m} \partial_x \phi^*(x) \partial_x \phi(x) + \mu \phi(x)^2 \ight.}$$

$$\left. - \frac{V_0}{\alpha} |\phi|^4 \right]$$

we can also add term of interaction in the action $\int d^d x \ d^d y \ \frac{1}{2} \phi^*(x) \ \phi(y) \ \phi(y) \ \phi(x)$

By using the short-range interaction approximation,

$$Z = \int D[\phi^*(x,t)] D[\phi(x,t)] \ e^{i S}, \ \text{where}$$

$$S = \int d^d x \ dt \ \left[ i \frac{1}{2} (\phi^* \partial_t \phi - \phi \partial_t \phi^*) - \frac{1}{2m} \partial_x \phi^* \partial_x \phi + \mu |\phi|^2$$

$$- \frac{V_0}{\alpha} |\phi|^4 \right]$$

The leading order approximation of thermal potential

$$V_2 = \int d^d x \ \frac{1}{2m} \partial_x \phi^* \partial_x \phi - \mu |\phi|^2 + \frac{V_0}{\alpha} |\phi|^4$$


5.2 symmetry breaking

A uniform solution \( \frac{\mathcal{V}_0}{V} = -\mu |\phi_0|^2 + \frac{V_0}{2} |\phi_0|^4 \), minimizing \( \mathcal{V}_0 \)

\[ \Rightarrow \phi_0 = \sqrt{\frac{\mu}{V_0}} e^{i\theta} \text{ for } \mu > 0 \]
\[ 0 \text{ for } \mu \leq 0 \]

\[ \Rightarrow \mathcal{V}_0(\mu) = \begin{cases} \frac{\mu^2}{V_0} & \mu > 0 \\ 0 & \mu \leq 0 \end{cases} \]

**U(1) symmetry**: \( \phi \rightarrow e^{i\theta} \phi \), \( S \) does not change.

However, the classical ground state has a fixed value of \( \theta \), which breaks the U(1) symmetry. A phase transition is related to a singularity of the thermo-dynamic potential, v.s. \( \mu \).

\( \phi_0 \rightarrow -\phi_0 \) symmetry

First order

Second order phase transition breaking symmetry
Long-range order \( \langle \Phi^+(x,t) \Phi(x,0) \rangle \bigg|_{(x,t) \to \infty} = 1 \Phi_0^2 \neq 0 \)

The symmetry breaking phase can be either represented by a nonzero order parameter, or a long-range correlation.

§ 3. Low energy effective theory

\[ \Phi = \Phi_0 + \delta \Phi \quad \text{at} \quad \mu < 0 \implies \Phi_0 \]

\[ S = \int d^4x \, dt \left( i \frac{1}{2} (\delta \Phi^* \partial \partial \partial \partial \partial \Phi - c.c.) - \frac{1}{2m} \partial_x \delta \Phi^* \partial_x \delta \Phi + \mu |\delta \Phi|^2 \right) \]

\[ \Rightarrow \quad \text{equation of motion:} \quad \left( i \partial_t \partial \partial \partial \partial \Phi - \frac{1}{2m} (-i \partial_x)^2 + \mu \right) \Phi = 0 \]

\[ \omega = \frac{1}{2m} k^2 - \mu, \quad \text{free boson.} \]

At symmetry-breaking ground state, we parameterize the

the ground state field configuration as \( \Phi = \sqrt{\rho_0 + \delta \rho} \, e^{i \Theta}, \rho_0 = \frac{\nu_0}{V_0} \)

\[ S = \int d^4x \, dt \left[ -(\rho_0 + \delta \rho) \partial \partial \partial \partial \partial \Theta - \frac{\rho_0 (\partial_x \Theta)^2}{2m} - \frac{(\partial_x \partial \rho)^2}{8m \rho_0} - \frac{V_0}{2} (\delta \rho)^2 \right] \]

(keep to quadratic order of \( \delta \rho \) and \( \Theta \)).

\( \Theta \) describes the slow, low-frequency mode, \( \delta \rho \) describes the massive fluctuations.
if we set $\delta p = 0$ \[ S = \int dx \, dt \left[ -p_x \dot{\theta} + \frac{1}{2m} p_x (\partial \times \partial \theta)^2 \right] \]

which is not correct, because $\partial \theta$ is a total derivative and does not enter equation of motion. We need to integrate out the massive field $\delta p$

\[ Z = \int D\theta \, D\delta p \, e^{i\int dx \, dt \left[ \frac{1}{2} \left( \frac{\partial \theta}{\sqrt{V_0 - \frac{\partial^2 x}{4m p_0}}} \right)^2 - \delta p \, \partial \theta + \frac{1}{2m} \delta p (\partial \times \partial \theta)^2 \right]} \]

integral over $\delta p$

\[ \approx \int D\theta \, e^{i\int dx \, dt \left[ \frac{1}{2} \left( \frac{\partial \theta}{\sqrt{V_0 - \frac{\partial^2 x}{4m p_0}}} \right) - \delta p \, \partial \theta \right]} \]

\[ \approx \text{const.} \, e^{i\int dx \, dt \, \frac{1}{2} \left( \partial \theta \right) \left( \frac{1}{\sqrt{V_0 - \frac{\partial^2 x}{4m p_0}}} \right)} \]

\[ Z = \int D\theta \, e^{i\int dx \, dt \, \frac{1}{2V_0} \left( \partial \theta \right)^2 - \frac{p_x}{2m} (\partial \times \partial \theta)^2} \]

$\theta$ lives on a circle, $\theta$ and $\theta + 2\pi$ refer to the same point.

we can introduce $Z = e^{i\theta}$ \[ \Rightarrow S_{eff} = \int dx \, dt \left\{ \frac{1}{2V_0} \left| \partial \theta \right|^2 - \frac{p_x}{2m} (\partial \times \partial \theta)^2 \right\} \]

$x$-$y$ model
equation of motion: \( \Theta(x,t) = \ddot{\Theta} + \beta \Theta \Rightarrow \)

\[
(-\frac{1}{2V_0} \frac{\partial^2}{\partial t^2} + \frac{P_0}{2m} \frac{\partial^2}{\partial x^2}) \Theta = 0 \Rightarrow \omega = \nu(k) \text{ with } \nu^2 = \frac{P_0 V_0}{m}.
\]

Density & current:

we add source term \(-A_0 \rho = -A_0 (\rho + \partial \phi), \quad \vec{A} \cdot \vec{j} = \vec{A} \cdot \text{Re}(\phi^\dagger \frac{\partial}{\partial t} \phi) = \vec{A} \frac{P_0}{m} \nabla \Theta \)

\[
\int d^3x d\tau \left( \frac{\partial}{\partial \tau} \left( \frac{\partial \Theta + A_0}{V_0} \right) \right) \frac{1}{V_0} \exp \left[ i \int d^3x d\tau \frac{1}{\nu V_0} (\partial \Theta + A_0)^2 \right]
\]

\[\Rightarrow \frac{1}{V_0} \frac{\partial \Theta \cdot A_0}{A_0 \rho} \Rightarrow \begin{bmatrix} \rho = \rho_0 - \frac{\partial \Theta}{V_0} \\ \vec{j} = \frac{P_0}{m} \nabla \Theta \end{bmatrix}
\]

§ phonon mode

In the above discussion the \( \Theta \) is considered as waves. According to the particle-wave duality, we can also consider them as bosons.

\[
\vec{\rho}_{\text{eff}} = \sum_k \frac{A_k}{2} \Theta_k \dot{\Theta}_k - \frac{B_k}{2} \Theta_k \Theta_k, \quad A_k = V_0^{-1}, \quad B_k = \frac{P_0 k^2}{m}
\]

\[
\Pi_k = \frac{\partial \vec{\rho}_{\text{eff}}}{\partial \dot{\Theta}_k} = A_k \dot{\Theta}_{-k} \Rightarrow H = \sum_k \left( \frac{1}{2A_k} \Pi_k \Pi_{-k} + \frac{B_k}{2} \Theta_k \Theta_{-k} \right)
\]

introduce \( \chi_k = \Theta_k \Theta_{-k} + i \Pi_k \Pi_{-k}, \quad \nu_k = \frac{1}{\sqrt{2}} (A_k B_k)^{1/4}, \quad \nu_\chi = \frac{1}{\sqrt{2}} (A_k B_k)^{-1/4} \)

we get \( H = \sum_k \varepsilon_k \chi_k \chi^*_k \)
Goldstone modes & spontaneous symmetry breaking

U(1) symmetry breaking $q \rightarrow q\, e^{i\theta}$, but the ground state does not have such a symmetry. Nambu, Goldstone proved that there's a gapless mode, which describes that the system is trying to restore the symmetry, i.e., the fluctuations among degenerate ground states.

Different degenerate ground states are related by symmetry transformation.

One can write low energy theory in terms of Goldstone modes, which is usually called non-linear $\sigma$ model.

1. Spontaneous symmetry can only happen in infinite systems. At finite systems, quantum fluctuations will restore the full symmetry. But for infinite system, it takes an infinitely long time to fluctuate from one state with one value of order parameter to another state by a different order parameter.

For example, for the superfluid boson state $q_0 = \langle \omega | \alpha \rangle \langle \alpha | \omega \rangle = l q \, e^{i\theta}$.

Let us define $W = e^{iN\theta}$, we know $[W, H] = 0$. 
\( W a(x) W^\dagger = a(x) e^{i\theta} \), \( \Rightarrow \) For finite system, \( |\psi\rangle \) is \( \hat{N} \) eigenstate

\[ \langle \psi | a | \psi \rangle = \langle \psi | W a(x) W^\dagger | \psi \rangle = e^{i\theta} \langle \psi | a | \psi \rangle = 0. \]

Let us look at how the volume \( \rightarrow +\infty \), how the fluctuation slow down and lead to a symmetry breaking ground state. We only consider uniform

\[ L = V \cdot \left( \frac{1}{2M} \left( \frac{d \Theta}{dt} \right)^2 - \rho_0 \frac{d \Theta}{dt} + \mu \rho_0 - \frac{1}{2} V_0 \rho_0^2 \right), \]

which describe a particle moving on a circle with \( \Theta \in (0, 2\pi) \). The mass of the particle \( M = \frac{V}{V_0} \), and \( \rho = \frac{2\Theta}{V} \). \( L = M \frac{d \Theta}{dt} - \frac{\mu V}{V_0} \).

\[ H = P \frac{d \Theta}{dt} - L = \frac{P^2}{2M} + \frac{\mu V}{V_0} P. \quad \text{As} \quad V \rightarrow +\infty, \text{the particle becomes classical, and does not move i.e.} \quad \dot{\Theta} = 0. \]

But as \( V \) is finite, \( P \) is quantized with momentum \(-\hat{N} \) \( \Rightarrow \)

\[ E_n = \frac{n^2}{2M} - \frac{\mu V n}{V_0 M} = \frac{V_0 n^2}{2V} - \mu n. \]

The state with \( P = -\hat{N} = -\frac{\mu V}{V_0} \) has the minimum energy, which can be written as

\[ | \Phi_0 \rangle = \int d\Theta \ e^{-i\frac{\mu V}{V_0} \Theta} \left( a \Theta \right). \]

\( \hat{N} \) is the conjugate operator to \( \Theta \), \( \hat{N} = -P_0 + i \hbar \frac{d}{d\Theta} \). Thus \( \hat{N} \)-eigenstate cannot have a well defined \( \Theta \)-value.
But as $V \to +\infty$, the low lying states with different $\hat{N}$, are nearly degenerate

$$|\Theta\rangle = \sum_n e^{i\pi n^2} |n\rangle,$$

the gap vanishes as $\frac{1}{V}$.

- the symmetry breaking state.