Lect 10 Superconductivity

The superfluidity theory we have discussed is for neutral systems. How about for charged systems, say, superconductors?

The neutral boson Lagrangian

\[ L(\phi) = \frac{i}{2} (\phi^* \partial_t \phi - \phi \partial_t \phi^*) - \frac{1}{2m} \partial_x \phi^* \partial_x \phi + \mu |\phi|^2 - \frac{V_0}{2} |\phi|^4 \]

has the U(1) symmetry $\phi \rightarrow e^{i \lambda} \phi$, but it does not have the global symmetry $\phi(x,t) \rightarrow e^{i f(x,t)} \phi(x,t)$.

For charged bosons (Cooper pairs of electrons)

\[ L(\phi, A_\mu) = \frac{i}{2} (\phi^* (\partial_0 + iA_0) \phi - \phi (\partial_0 - iA_0) \phi^*) - \frac{1}{2m} |(\partial_i + iA_i) \phi|^2 \]

\[ + \mu |\phi|^2 - \frac{V_0}{2} |\phi|^4 + \frac{1}{8 \pi e^2} \left( \frac{1}{c} E^2 - CB \right) \]

where $E_i = \partial_i A_i - \partial_i A_0 = F_{0i}$, $B_i = \varepsilon_{ijk} \partial_j A_k = \frac{1}{2} \varepsilon_{ijk} F_{jk}$

which is invariant as

$\phi \rightarrow \tilde{\phi} = e^{i f(x,t)} \phi$, $\tilde{A}_\mu = A_\mu - \partial_\mu f$

If $\phi$, $A$ satisfy the classical equation of motion (fix $A_\mu$).

$\Rightarrow$ do variation respected to $\delta \phi$: $\phi' = e^{i f(x,t)} \phi \Rightarrow \delta \phi' = \phi (1 + i f(x,t))$

$\Rightarrow \delta L \left[ \tilde{\phi}, A_\mu \right] = \delta L \left[ \phi, A_\mu - \partial_\mu f \right]$ \hspace{1cm} $J_i = \frac{i}{2m} (\phi^* \partial_i \phi + \phi \partial_i \phi^*)$

$\Rightarrow \delta L = \int dx \, dt \, \partial_t \phi^* \cdot J^\mu \Rightarrow \partial_\mu J^\mu = 0 + A_i |\phi|^2$ \hspace{2cm} where $J_0 = \phi^* \phi$.
Current correlation function & E-M responses

\[ \mathcal{L}[\varphi, A_{\mu}] = \mathcal{L}[\varphi] - A_0 j^0 - A_i j^i - \frac{i}{2m} (A_i)^2 \rho \]

We use the linear response theory

\[ j_i = -\frac{i}{2m} [\varphi^* (2i \varphi) - (2i \varphi)^* \varphi] \]

\[ \langle j^\mu(x,t) \rangle = \langle j^\mu(x',t') \rangle + (1 - \partial_\mu) A_{\mu} \rho \]

\[ = \int dx'^{} d\tau' \Pi^{\mu \nu} (x,t; x', t') A_\nu (x', t') \]

The response function reads

\[ \Pi^{\mu \nu} (x,t; x', t') = -i \Theta (t-t') \left\langle [\rho(x,t), \rho(x', t')] \right\rangle \]

\[ \Pi^{0i} (x,t; x', t') = -i \Theta (t-t') \left\langle [\rho(x,t), j^i (x', t')] \right\rangle \]

\[ \Pi^{i0} (x,t; x', t') = -i \Theta (t-t') \left\langle [j^i (x,t), \rho(x', t')] \right\rangle \]

\[ \Pi^{ij} (x,t; x', t') = -i \Theta (t-t') \left\langle [j^i (x,t), j^j (x', t')] \right\rangle + \delta^{ij} \int d(x-x') d(t-t') \frac{\rho}{m} \]

\(\Pi^{\mu \nu}\) satisfies

\[ (\Pi^{\mu \nu}(x,t; x', t'))^* = \Pi^{\nu \mu}(x', t'; x, t) \]

\[ \Rightarrow (\Pi^{\mu \nu}(k))^* = \Pi^{\nu \mu}(-k) \]

\(\Pi^{\mu \nu}\) satisfies continuity & gauge invariance conditions.

\[ \exists M J^\mu = 0 \Rightarrow k_\mu \Pi^{\mu \nu}(k) = 0 \]

\[ A \rightarrow A + e \phi \Rightarrow \Pi^{\mu \nu}(k) k_\nu = 0 \]
we can decompose \( \mathbf{\Pi}^j \) into transverse & longitudinal parts

\[
\mathbf{\Pi}^j (k) = \frac{k_i k_j}{k^2} \mathbf{\Pi}^z (k) + (\delta^j_i - \frac{k_i k_j}{k^2}) \mathbf{\Pi}^\perp (k)
\]

spatial part

\[
\mathbf{\Pi}^0 (k) = (\mathbf{\Pi}^0 (-k)^\dagger = -\frac{k_j}{\omega} \mathbf{\Pi}^j (k) \implies k \mu \mathbf{\Pi}^\mu (k) = 0
\]

\[
\mathbf{\Pi}^{00} (k) = -\frac{k_j}{\omega} \mathbf{\Pi}^{0j} = \frac{k_i k_j}{\omega^2} \mathbf{\Pi}^j (k)
\]

\[
\begin{align*}
\mathbf{\Pi}^{0i} (k) &= -k_i \mathbf{\Pi}^z (k) \\
\mathbf{\Pi}^{00} (k) &= \frac{k^2}{\omega^2} \mathbf{\Pi}^z (k)
\end{align*}
\]

the time component is only related to the longitudinal component.

For the correlation in the spatial direction, we define the paramagnetic contribution as

\[
\mathbf{\Pi}_{\text{para}}^{ij}(x, t; x', t') = -i \Theta (t - t') \langle \mathbb{E} j^i (x, t), j^j (x', t') \rangle,
\]

\[
\begin{align*}
\mathbf{\Pi}^z (k) &= \mathbf{\Pi}_{\text{para}}^z (k) + \frac{\langle p \rangle}{m} \\
\mathbf{\Pi}^\perp (k) &= \mathbf{\Pi}_{\text{para}}^\perp (k) + \frac{\langle p \rangle}{m}
\end{align*}
\]

diamagnetic contribution

we know \( \mathcal{P}^0 (k) = \mathbf{\Pi}^{00} (k) A^0 (k) \).

Since \( A^0 (k) \) is just the external potential, thus \( -\mathbf{\Pi}^{00} (k) \) is just the compressibility \( \chi (k) = -\mathbf{\Pi}^{00} (k) \). Let us assume that \( \chi (k) \rightarrow \text{const} \ (k \rightarrow 0) \).
then \( \Pi''(k) = -\chi(k) \frac{\omega^2}{k^2} \), thus at the static limit (\( \omega \to 0 \) first; \( k \to 0 \) second), \( \Pi''(k) \) goes to zero (longitudinal field has no effect!), thus \( \Pi_{\text{para}}(k) \) must exactly cancel the diamagnetic contribution.

On the other hand, if \( \omega \to 0 \) first and \( k \to 0 \) second (\( k \ll \omega \)), \(-iwA(k,w)\) is the electric field

\[
J_i(\omega) = \lim_{k \to 0} \frac{\Pi_{ij}}{-iw} \frac{(-iw)A_j(k,w)}{E(\omega,k)}
\]

\[
\Rightarrow \quad \sigma(\omega) = \frac{\Pi''(\omega,0)}{-iw} = \frac{\Pi(\omega,0)}{-iw}
\]

\( \sigma \rho = 4\pi \rho \) \( \Rightarrow \) \( \vec{D} = \vec{E} + 4\pi \vec{P} = (1+4\pi \chi)\vec{E} = \vec{E}
\)

\( \vec{j} = \mathbf{2} \rho \) \( \Rightarrow \) \( \vec{j}(\omega) = -i\omega \rho(\omega) = -i\omega \chi(\omega) \vec{E}
\)

\( \vec{j}(\omega) = \frac{1}{4\pi} (\mathbf{E} - 1) \omega \vec{E}(\omega) \)

i.e. \( \sigma(\omega) = \frac{-i}{4\pi} (\mathbf{E} - 1) \omega \quad \text{i.e.} \quad \mathbf{E}(\omega) = 1 + \frac{i}{4\pi \sigma(\omega)} \)
how about in the static limit $w \ll k$, $\Pi_{\perp}(k, w)$?

From $\nabla \times \vec{M} = \vec{J}$

$$\Rightarrow \dot{j}_i = \epsilon^{ijkl} \nabla_{j} M_k = -\Pi_{ij} A_k = - \epsilon^{ijkl} \epsilon^{kl'm} A_j \frac{\Pi_{\perp}(k, w)}{k^2}$$

$$= \epsilon^{ij'k'} \epsilon^{k'lj} \epsilon^{k'lij} k_{j'} A_j \frac{\Pi_{\perp}(k, w)}{k^2} = - \epsilon^{ijk} B_k \frac{\Pi_{\perp}(k, w)}{k^2}$$

$$\Rightarrow M_i = -\frac{\Pi_{\perp}(k, 0)}{k^2} B_i$$, i.e. $\frac{\Pi_{\perp}(k, 0)}{k^2}$ is magnetic susceptibility.

Summarize: Compressibility $\lim_{k \to 0} \chi(k) = \lim_{k \to 0} \frac{k^2}{\dim} \lim_{\omega \to 0} \frac{\Pi_{\|}(k, \omega)}{\omega^2}$

orbital magnetic susceptibility $\lim_{k \to 0} \chi_{\text{m}}(k) = \lim_{k \to 0} \frac{1}{k^2} \Pi_{\perp}(k, 0)$

* E-M:

let us calculate $\Pi_{\|}$ for bosonic field:

$$L = -\frac{1}{2} \left( \partial_0 \Theta + A_0 \right)^2 - \frac{1}{2} \left( \partial_i \Theta + A_i \right)^2$$

the paramagnetic current $\dot{j}_0 = -\chi \partial_0 \Theta$, $\dot{j}_i = \chi \omega^2 \partial_i \Theta$

total current $J_0 = -\chi (\partial_0 \Theta + A_0)$, $J_i = \chi \omega^2 (\partial_i \Theta + A_i)$
\[ \Pi_{\text{para}}^{00}(k,w) = \chi^2 (-iw)(+iw) \langle \Theta(wk)\Theta(-w,-k) \rangle = \frac{\chi^2 \omega^2}{\omega^2 - v^2 k^2 + i0^+ \text{sgn}(w)} \]

\[ \Pi_{\text{para}}^{0i}(k,w) = \Pi_{\text{para}}^{i0}(k,w) = \chi^2 (-iw)(-ik_i) \langle \Theta(wk)\Theta(-w,-k) \rangle = \frac{\chi^2 v \delta_i}{\omega^2 - v^2 k^2 + i0^+ \text{sgn}(w)} \]

\[ \Pi^{ij} = \chi^2 (ik_i)(-ik_j) \langle \Theta(wk)\Theta(-w,-k) \rangle = \chi v^2 \frac{\delta_{ij} k_i k_j}{\omega^2 - v^2 k^2 + i0^+ \text{sgn}(w)} \]

\[ \Pi^{00}(k,w) = \chi \left[ \frac{\omega^2}{\omega^2 - v^2 k^2 + i0^+ \text{sgn}(w)} - 1 \right] = \chi \left[ \frac{\omega k_i}{\omega^2 - v^2 k^2 + i0^+ \text{sgn}(w)} \right] \]

\[ \Pi^{0i}(k,w) = \Pi^{i0}(k,w) = -\chi v^2 \frac{\omega k_i}{\omega^2 - v^2 k^2 + i0^+ \text{sgn}(w)} \]

\[ \Pi^{ij}(k,w) = \chi v^2 \left( \delta_{ij} + \frac{v^2 k_i k_j}{\omega^2 - v^2 k^2 + i0^+ \text{sgn}(w)} \right) \]

\[ \Pi^{10} = \chi v^2 \]

\[ \lim_{k \to 0} \lim_{w \to 0} -\Pi^{00}(k,w) = \lim_{k \to 0} -\Pi^{10}(k,0) = \chi. \]

\[ \text{the Compressibility} \]

\[ \text{magnetic susceptibility} \]

\[ \lim_{k \to 0} \frac{1}{k^2} \Pi^{\perp}(k,0) \]

\[ \text{if we choose transverse gauge } \partial_i A_i = 0 \]

\[ \Rightarrow J^i = \Pi^{ij} A_j = \Pi^{\perp} A_i = \chi v^2 A_i + \frac{p}{m} A_i \]

\[ \text{Lorenz equation!} \]
optical conductivity \[ \Re \sigma(w) = -\Im \frac{\Pi''(w,0)}{w} = \Im \frac{\chi v^2}{\omega + i0^+} = \frac{\pi \rho \delta(w)}{m} \]

which is zero at finite frequency.

* Anderson - Higgs mechanism

In charged superfluid (superconductor), the gapless Goldstone mode is gone. The gauge field \( A_\mu \) has its own dynamics.

\[ \mathcal{L} = \int d^4x dt \, L(\varphi, A_\mu) + \frac{1}{8\pi^2} (E^2 - B^2) \]

in the symmetry breaking ground states \( \varphi = |\varphi| e^{i\theta(x,t)} \)

we can absorb the phase by doing a gauge transformation

\[
\left| (\partial_\mu - iA_\mu) |\varphi| e^{i\theta(x,t)} \right|^2 = \left| (\partial_\mu - iA_\mu - i\tilde{\theta}) |\varphi| \right|^2
\]

\[ i\tilde{A}_\mu \] which is equivalent to \( iA_\mu \), and does not give new physical field configuration.

\[ \Rightarrow \ \\
\text{or equivalently we can set } \theta = 0 \text{ in the action } \]

\[ \mathcal{L} = \frac{\chi}{a} \left( (\partial_\mu \Theta + A_\mu)^2 - v^2 (\partial_\mu \Theta + A_\mu)^2 \right) = \frac{1}{2} A_\mu^2 - \frac{\rho}{2m} A_\mu^2 \]
The Lagrangian for $A$ becomes

$$L = \frac{1}{2\nu_0} A_0^2 - \frac{\rho}{2m} A_i^2 + \frac{1}{8\pi e^2} (E^2 - B^2)$$

note that

$$\frac{1}{8\pi e^2} E^2 = \frac{1}{8\pi e^2} [\partial_0 A_i]^2 + \frac{1}{8\pi e^2} (\partial_i A_0)^2 - \frac{1}{4\pi e^2} \partial_i A_0 \partial_0 A_i$$

and thus $A_0$ has no time derivative. Let's integrate out $A_0$.

$$A_0 \left( \frac{1}{2\nu_0} - \frac{\partial_i^2}{8\pi e^2} \right) A_0 + \frac{1}{4\pi e^2} A_0 \partial_0 \partial_i A_i \quad \text{neglected}$$

$$\approx \frac{1}{2\nu_0} (A_0 - \frac{\nu_0}{4\pi e^2} \partial_0 \partial_i A_i)^2 - \frac{1}{2\nu_0} \frac{\nu_0^2}{(4\pi e^2)^2} (\partial_0 \partial_i A_i)(\partial_0 \partial_j A_j)$$

(const after gaussian integration)

$$\Rightarrow \quad L_{\text{eff}} = \partial_0 A_i \left[ \frac{1}{8\pi e^2} \delta_{ij} + \frac{1}{2} \frac{\nu_0}{(4\pi e^2)^2} \partial_i \partial_j \right] \partial_0 A_j - \frac{\rho}{2m} A_i^2 - \frac{B^2}{8\pi e^2}$$

Let us decompose $A_i = \hat{k}_i A'' + \hat{n}_i A_\perp$

$$L_{\text{eff}} = \frac{1}{8\pi e^2} \left[ (\partial_0 A_\perp)^2 - (\partial_i A_\perp)^2 \right] - \frac{\rho}{2m} (A_\perp)^2$$

$$+ \frac{1}{8\pi e^2} \partial_0 A'' \left( 1 + \frac{\nu_0}{\omega e^2} (\partial_i)^2 \right) \partial_0 A'' - \frac{\rho}{2m} (A''^2)^2$$

- all the 3 modes has the gap $\Delta = e\sqrt{4\pi\rho/m}$

$$\Rightarrow \quad L_{\text{eff}} = \frac{1}{8\pi e^2} \left[ (\partial_0 A_\perp)^2 - (\partial_i A_\perp)^2 \right] - \frac{\rho}{2m} (A_\perp)^2 + \frac{1}{8\pi e^2} \left[ \partial_0 A'')^2 - (\partial_i A'')^2 \right] - \frac{\rho}{2m} (A''^2)^2$$
Superfluidity & superfluid density:

Let's consider a boson system which is invariant under Galileo transformation, and assume an excititation spectrum $E(k)$. Consider a single excititation $E_t(k)$, thus $p^t = \frac{p}{c}$ and $E = E_{ground} + E_t$. Let us boost the system by velocity $\vec{v}$, thus excititation has additional total energy and momentum change to

$$E = E_{ground} + E + \frac{1}{2} N m v^2 + \vec{v} \cdot \vec{k}$$
$$\vec{p} = \vec{k} + N m \vec{v}$$

Thus compared with boosted ground state, we have

$$E_{v(k)} = E(k) + \vec{k} \cdot \vec{v}, \quad \vec{p} = \vec{k}.$$

In the boosted superfluid, in the equilibrium, we have distribution according to

$$E_v = E_g + \sum_k E_v(k) n_b(E_v(k)) + \frac{1}{2} N m v^2$$
$$\vec{p} = N m \vec{v} + \sum_k \vec{k} n_b(E_v(k))$$

The momentum can be written as, (expand to first order of $\vec{v} \cdot \vec{k}$).

$$\vec{p} = (N m - V \rho_n m) \vec{v} = V \rho_s m \vec{v}$$

↑ with superfluid density.
where \( \rho_n = -\frac{1}{md} \int \frac{dk}{(2\pi)^d} k^2 N_{e'}(E(k)) \).

If \( \langle \vec{v}, \vec{k} \rangle \approx \langle k \rangle \), then the integral can be averaged.

\[ \Rightarrow \rho_n \propto T^{d+1} \]

Another question is: Since super-current carrying state is not the lowest energy state, why is it stable?

Let us consider a symmetry breaking state \( \Phi = \Phi_0 + \delta \Phi \uparrow \) condensate fluctuation

Let us twist \( \Phi_0 \) (boost):

\[ \Phi' = \Phi_0 e^{imv_0 x} \] such that \( m v_0 L = 2\pi n \).

Because \( |\Phi_0| \) is fixed, we cannot untwist the condensate without suppressing \( |\Phi_0| \) to zero. Thus although \( \Phi_0 e^{imv_0 x} \) has a high energy, it cannot relax unless vortex tunneling.

The vortex tunneling is the decaying mechanism of superfluidity.

Superfluid won't flow forever, but it is long-lived!

Create a vortex in the inner wall and move it across the sample to outer wall untwist the superfluid...