Lect 8. Superfluid at finite temperatures, low dimensions

We will use imaginary path integral

\[ \mathcal{Z} = \int D\Phi \exp \left[ -\int_0^\beta d^4x \left( \frac{1}{\lambda} (\Phi^* \partial_\mu \Phi - \Phi \partial_\mu \Phi^*) + \frac{1}{2m} \partial_\mu \Phi^* \partial_\mu \Phi - \mu 1 \Phi^2 \right) \right. \\
\left. + \frac{\lambda_0}{2} |\Phi|^4 \right] \]

in the time domain, we have a finite size \( \beta = \frac{1}{k_B T} \), thus if the correlation length in the time domain \( \xi_T \) is much larger than \( \beta \), then we can neglect the fluctuation in the T-domain. (The relation between \( \xi_T \sim \xi_0^2 \), where \( \xi_0 \) is called dynamic critical exponent.) In this case, we arrive at the

classic partition function

\[ \mathcal{Z} = \int D\Phi \exp \left[ -\beta \int d^4x \left( \frac{1}{2m} |\partial_\mu \Phi|^2 \right) \right] \text{, where only phase fluctuations are kept.} \]

\[ \Rightarrow \text{Seff} = \int d^4x \frac{\beta}{2} (\partial_\mu \Theta)^2 \text{, } \eta = \frac{1901^2}{mT_k^3} \text{, } T_k \text{ is the phase rigidity.} \]

Another important question is whether long range order can survive under thermal fluctuations.

Using the result in the last lecture, we have at \( d \geq 3 \), thermal fluctuations does not always destroy long range order.
For $d < 2$, thermal fluctuations always destroy superfluidity:

$$Z[J] = \int \mathcal{D}\theta \ e^{-\frac{\alpha}{2} \int d^d x \ J(x) \ \theta(x)} \ \text{where} \ J(x) = \delta(x) - \delta(x_0)$$

$$\langle e^{i\theta(x)} e^{-i\theta(x')} \rangle = \frac{Z[J]}{Z[0]} = \frac{1}{Z[0]} \int \mathcal{D}\theta \ \exp\left\{ -\frac{1}{\alpha} \int d^d x \ \theta(x) \ G(x, x') \ \theta(x') + iJ(x) \ \theta(x) \right\}$$

$$= \exp\left\{ -\frac{1}{\alpha} \int d^d x \ d^d x' \ J(x) \ G(x, x') \ J(x') \right\}, \ \text{where} \ G \ \text{is the inverse of} \ -\partial_x^2$$

$$\langle \theta(x_1) \ \theta(x_2) \rangle = \int \mathcal{D}\theta \ \theta(x_1) \ \theta(x_2) \ e^{-\frac{1}{\alpha} \int d^d x \ \theta(x) \ G(x, x') \ \theta(x')} \ \frac{1}{Z[0]} \int \mathcal{D}\theta \ \exp\left\{ -\frac{1}{\alpha} \int d^d x \ \theta(x) \ G(x, x') \ \theta(x') + iJ(x) \ \theta(x) \right\}$$

$$= G(x_1, x_2)$$

$$\langle e^{i\theta(x)} e^{-i\theta(x')} \rangle = e^{\frac{1}{\alpha} \int d^d x \ d^d x' \ J(x) \ \theta(x) \ G(x, x') \ \theta(x')} = e^{\langle \theta(x) \ \theta(x') \rangle - \langle \theta(x) \ \theta(x') \rangle}$$

$$\langle \theta(x) \ \theta(x) \rangle = -\frac{1}{\alpha} \int d^d k \ \frac{1 - e^{i k \cdot x}}{k^2}$$

$$= \begin{cases} 
-\frac{1}{\alpha} \left[ \frac{K_d \ \Lambda^{d-2}}{(d-2)} \right] & \text{for } x \to \infty \ (d > 2) \\
\frac{-1}{\alpha} \frac{1}{\alpha^{d-1}} \ln\left( \frac{\Lambda}{\Lambda - 1} \right) & \text{for } x \to \infty \ (d = 2) - \text{the wave vector cut off} \\
\frac{-1}{\alpha} \frac{1}{\alpha} |\chi| & \text{for } x \to \infty \ (d = 1) 
\end{cases}$$
at $d = 1 \Rightarrow \langle e^{i\theta(x)} e^{-i\theta(x)} \rangle \sim e^{-\frac{1}{2\pi} \ln |x|}$, which is exponentially decaying, with decay length $\frac{1}{2\pi}$.

at $d = 2 \Rightarrow \langle e^{i\theta(x)} e^{-i\theta(x)} \rangle \sim e^{-\frac{1}{2\pi\eta} \ln \frac{|x|}{\Lambda^{-1}}} = \left( \frac{\Lambda^{-1}}{|x|} \right)^{\frac{1}{2\pi\eta}}$

which has power-law decaying, with exponent $\frac{1}{2\pi\eta}$

for $d > 2$, there's no infra-real divergence.


So far we have completely neglected the compactness of $\theta$, we will see at $2D$, it gives rise to the topological excitation of vortices, and its effect to change the transition to K-T type.

The action of a single vortex: $\Phi(x, y) = \Phi_0$ for $r \to 0$ and $\Phi_0 \to \infty$

$S_v = \int d^2 r \frac{1}{2} \nabla^2 \frac{1}{r} + S_c = \pi \eta \ln \frac{L}{\xi} + S_c \leftarrow$ core energy

The interaction between two vortices can be calculated by the analogy with 2D electro-statics.

$\frac{1}{2} \pi \nabla \Phi \nabla \Phi \leftrightarrow \frac{E^2}{8\pi} \Rightarrow E = \sqrt{4\pi \eta} \nabla \Phi = \frac{2\pi}{4\pi}$

$q = \frac{2\pi \cdot \sqrt{4\pi \eta}}{4\pi}$

$\Delta E = \sqrt{4\pi \eta} \int \frac{d r}{r} \cdot \nabla \Phi = \frac{2\pi \eta \ln \frac{r}{\xi}}{\sqrt{4\pi \eta}}$
To calculate the partition function, we need to include the vortex configuration. For a fixed vortex configuration \( \Phi_c = e^{i \Theta_c} \) and other fluctuations contribute as

\[
Z = \int D \Phi \ e^{- \frac{1}{2} \int dx \ \left( \frac{\partial}{\partial x} (\Theta_c + \Phi) \right)^2} = \int D \Phi \ e^{- \frac{1}{2} \int dx \ \left( \frac{\partial}{\partial x} \Theta_c \right)^2 + \int dx \ \frac{1}{2} \left( \frac{\partial}{\partial x} \Phi \right)^2 + \int \Phi \left( \frac{\partial}{\partial x} \Theta_c \right) \Phi \ dx}
\]

\[
= e^{- S_{\text{eff}}(\Theta_c)} \int D \Phi \ e^{- \frac{1}{2} \int dx \ \left( \frac{\partial}{\partial x} \Phi \right)^2 + \int dx \ \frac{1}{2} \left( \frac{\partial}{\partial x} \Theta_c \right)^2} \leq 0
\]

\[
= e^{- S_{\text{eff}}(\Theta_c)} \cdot Z_0 \quad \text{contribution from vortex free configuration}
\]

given by the long-range interaction between vortices.

\[
Z = \frac{1}{2 \pi} \sum_{n} \frac{1}{n! n!} \int_{2 \pi} d^2 \mathbf{r}_c \ \bar{e}^{2 \pi n \Theta_c} \ \epsilon \sum_{i,j} (2 \pi n) \ \epsilon \ |i,j| \ \ln \frac{R_{ij}}{\ell}
\]

we must have equal number of vortices and anti-vortices, to avoid energy divergence. \( |i,j| = \pm 1 \), satisfying \( \sum |i| = 0 \).

Let us estimate the effective action \( Z = e^{S_{\text{eff}}} \).

\[
S_{\text{eff}} \sim 2 \pi n \ln n + n \left( 2 \pi \ln \frac{\ln \frac{\ell}{\ell}}{\ell} + 2 \mathbb{S}_c \right) - 2 \pi n \ln \frac{L^2}{\ell^2}
\]

\[
= \frac{L^2}{\ell^2} \cdot \frac{2}{n} \left( (2 \pi - 2) \ln \frac{\ell}{\ell} + \mathbb{S}_c \right)
\]

where \( \ln \) is the average inter-vortex distance.
let us plot $S_{\text{eff}}$ vs $l/\ln l$.

If $S_c \ll -1$, thus it's cheap to create vortices, we minimize $S_{\text{eff}}$

$$S_{\text{eff}} = 2(\frac{d}{c})^2 \cdot \left(\frac{l}{\epsilon}\right)^2 \left[ S_c - (\zeta \pi - 2) \ln \frac{l}{\epsilon} \right]$$

if $\zeta \pi < 2$,

$$> \pi$$

$\Rightarrow$ proliferation of vortices, $\ln l \to l$

$$\langle e^{i\phi(x)} e^{-i\phi(y)} \rangle \sim e^{-1(l)^{1/3}}$$, where $\zeta \to l$.

if $S_c > 1$, it's expensive to make vortices.

if $\zeta \pi < 2$, high temperature

the optimal $l/\ln l \sim e^{-S_c/(2-\zeta \pi)}$

i.e. $\ln l \sim e^{2-\zeta \pi}/l$, a finite number of vortices.

if $\zeta \pi > 2$, low temperature $T < \frac{\pi}{2} \rho_s$

no free vortices.
Thus when vortices are expensive (Σc > 1), we have a transition as increasing temperature.

\[ \frac{\mathcal{E}_c}{\pi} \rightarrow +\infty \]

\[ S \sim \ln l \sim l \exp \left( \frac{\mathcal{E}_c}{T} \right) \]

\[ \eta = \frac{\mathcal{E}_c}{T} \sim \frac{2}{\pi} \]

\[ T_k = \frac{\pi}{\mathcal{E}_c} \]

Naively, we would expect:

\[ S \sim l \exp \left( \frac{\mathcal{E}_c}{T} \right) \sim l \exp \left( \frac{\mathcal{E}_c T_{k/3}}{T - T_{k/3} \mathcal{E}_c} \right) \]

But as \( T \rightarrow T_k \), \( \mathcal{E}_c \) changes. Actually, \( 2 - \mathcal{E}_c T_{k/3} \sim (T - T_{k/3})^{1/2} \)

And \( S \sim \pi l \exp \left( \frac{\mathcal{E}_c}{T - T_{k/3}} \right)^{1/2} \) (We will analyze it later).

Chapter 8: Renormalization group & Scaling dimensions

Relevant perturbation can change the long distance behavior of the system. How to decide a perturbation is relevant or not can be learned from the scaling dimension analysis.

Say \( \varepsilon \mathcal{E}_c \ll 1 \), i.e. it seems that vortex is costly. But we know at \( 2\pi < 2 \), no matter how small \( \varepsilon \mathcal{E}_c \) is, vortex will destroy
the algebraic correlation of \( \langle e^{iQ(x)} e^{iQ(y)} \rangle \). Then it's a relevant perturbation. On the other hand, at \( \eta \pi > 2 \), vortices is an irrelevant perturbation.

Let us consider a theory of \( \mathcal{S} = \mathcal{S}_0 + \int d^d x \frac{g}{2} \mathcal{O}(x) \), in which

\[
\langle \mathcal{O}(x) \mathcal{O}(y) \rangle \sim \frac{1}{|x-y|^{2\Delta}}
\]

where \( \ell \) is the short energy length scale.

At second order perturbation

\[
\Delta \mathcal{S}_{\text{eff}} = -\ln \mathcal{Z} + \ln \mathcal{Z}_0 = \ln \left[ \int \frac{d^d \mathbf{k}}{(2\pi)^d} \int d^d x \int d^d y \frac{g^2}{\ell^\Delta} \langle \mathcal{O}(x) \mathcal{O}(y) \rangle \right]
\]

\[
= -2\ln g - \ln \frac{\int \frac{d^d \mathbf{k}}{(2\pi)^d} \int d^d x \int d^d y}{\ell^\Delta}
\]

\[
= -2\ln g - \left[ \ln \left( \frac{L}{\ell} \right)^d + \ln \left( \frac{L}{\ell} \right)^{d-2\Delta} \right] = -2\ln g - 2(d-\Delta) \ln \frac{L}{\ell}
\]

if \( \Delta < d \), \( \Delta \mathcal{S}_{\text{eff}} < 0 \), the system prefers to have two \( \mathcal{O} \)-operators appear at short-distance (at the order of \( \xi \)). Thus if we are
Interested at long distance behaviour. Perturbation of zero is always important. On the other hand if \( \Delta > d \), it's irrelevant for long range correlations.

\[ S = \int d^2 x \left[ \frac{K}{2} (\Delta \varphi)^2 - g \cos(\varphi) \right] \]

\( \mathbb{Z}_n \)-symmetry
\( \varphi \to \varphi + \frac{2\pi}{n} \), hence \( \varphi \) is non-compact.

The duality between 2D \( \text{xy} \)-model and the 2D clock model

Let us consider a generic clock model

\[ Z = \int D \varphi \ e^{\int d^2 x \left[ \frac{K}{2} (\Delta \varphi)^2 - g \cos(\varphi) \right]} \]

\[ = \int D \varphi \ e^{-\int d^2 x \left[ \frac{K}{2} (\Delta \varphi)^2 \right]} \frac{\prod_{k} \left[ \int d^2 x \ e^{i\theta_k + e^{-i\theta_k}} \right]^k}{(2\pi)^k} \]

\[ = Z_0 \sum_k \frac{1}{k! k!} \prod_{i=1}^{2k} \left( \frac{d^2 x}{2\pi} \right)^k \]

… how to calculate \( \langle e^{i\sum_{k=1}^n \theta_k} - \frac{4k}{\pi} \theta_k \rangle \).
Again, we introduce field \( J(x) = \sum_{i=1}^{K} \delta(x-x_i) \) as a source

\[
\langle e^{i \sum_{i=1}^{K} \Theta(x-x_i) - \sum_{i=K+1}^{2K} \Theta(x-x_i)} \rangle = \exp \left[ -\frac{1}{\alpha} \int dx dx' J(x) G(x-x') J(x') \right],
\]

where \( G \) is the inverse of \(-\alpha J^2\).

\[
= \exp \left[ -\sum_{i<j} q_{i,j} \langle \Theta(x_i) \Theta(x_j) \rangle - \frac{1}{\alpha} \sum_{i=1}^{K} q_i^2 \langle \Theta(x_i) \Theta(x_i) \rangle \right]
\]

\[
= \exp \left[ +\sum_{i<j} \frac{q_{i,j}}{2\pi \alpha x} \ln \frac{|x_i-x_j|}{L} \right] \cdot \exp \left[ +\frac{1}{\alpha} \sum_{i=1}^{K} \frac{q_i^2}{2\pi x} \ln \frac{L}{L} \right]
\]

\[
= \exp \left[ +\sum_{i<j} \frac{q_{i,j}}{2\pi x} \ln \frac{|x_i-x_j|}{L} \right] \cdot \exp \left[ -\frac{1}{\alpha} \sum_{i=1}^{K} \frac{q_i^2}{2\pi x} \ln \frac{L}{L} \right]
\]

\[
= \left[ \frac{|x_i-x_j|}{L} \right] + \frac{q_{i,j}}{2\pi x}
\]

\[
\Rightarrow z = \frac{z_0}{\pi} \sum_{i<j} \frac{1}{k! k!} \int \frac{2\pi}{i=1} e^{-2\pi x^2} \sum_{i<j} \frac{q_{i,j}}{2\pi x} \ln \frac{r_{ij}}{L}
\]

which is the same as vortex partition function, if

\[
e^{-S_c} = \frac{g}{2}, \quad \text{and} \quad 2\pi \eta = \frac{1}{2\pi x}.
\]

Then the vortex in the \( x_y \) model maps into \( e^{i \Phi(x)} \) operator in the \( \Phi \) model.
xy model

\[ S = \int \frac{1}{2} \left( \partial x \Theta \right)^2 dx^2 \]

\[ 2\pi \phi = \frac{1}{2\pi} \frac{1}{k} \]

\[ \frac{1}{e} \phi = g/2 \]

\[ \text{compact, allow vortex } (\phi + 2\pi = \phi) \]

U(1) symmetry, no vertex operator

non-local excitation: vortex

\[ \leftrightarrow \quad \text{vertex operator} \quad e^{\pm i\phi} \]

\[ \text{vertex operator } (\text{local operator}) \]

How about compact clock model

\[ S = \int \frac{1}{2} \left( \partial x \Theta \right)^2 + g \cos \Theta \quad \leftrightarrow \quad S = \int \frac{k}{2} \left( \partial x \Phi \right)^2 + g \cos \Phi \]

\[ \text{vortex of the field } \Phi. \]

\[ \text{vortex fluctuating, } \Phi \text{ is also compact} \]

\[ \text{vortex fluctuating, in the original } xy \text{ model} \]
Lect 9: RG analysis to K-T transition

Let us consider the dual theory of XY model: the non-composite clock model

\[ S = \int d^2x \left( \frac{K}{2} \langle \Delta \phi \rangle^2 + g \cos n \phi \right) \]

where we set \( n \) to a general number.

as we know

\[ \langle e^{i n \phi} \rangle = (\frac{1}{|n|}) \frac{n^2}{2\pi K} \]

the value of the correlation function depends on the short length scale! In order to have a well-defined field theory, we must explicitly specify a short length scale! i.e. we would like to write

\[ S = \int d^2x \left( \frac{K}{2} \langle \Delta \phi \rangle^2 - \overline{g} \cos \Theta_k \right), \]

where \( \Theta_k(x) = \int d^2k \Theta_k e^{ikx} \)

we have three parameters \( K, \overline{g} \) and \( \frac{1}{2} \), and we can make two dimensionless parameters \( K_\perp \) and \( \frac{\overline{g}}{\frac{1}{2} \frac{1}{2}} \), small

we would think there are two different phases: At large \( K \) and small \( \overline{g} \), fluctuations are large and pinning potentials are weak, we have a \( \mathbb{Z}_n \)-symmetric state. On the other hand, if \( \overline{g} \) is large, pinning potentials are strong and fluctuations are weak, we have a symmetry breaking ground state.
Let us take the cosnθ term as perturbation. As we explained before, the scaling dimension \( \Delta = \frac{n^2}{4\pi X_\xi} \), thus at \( \frac{n^2}{4\pi X_\xi} \geq 2 \) (i.e. \( X_\xi < \frac{n^2}{8\pi} \)), the clock term is irrelevant; at \( \frac{n^2}{4\pi X_\xi} < 2 \) (i.e. \( X_\xi > \frac{n^2}{8\pi} \)), the clock term is relevant. Thus \( X_\xi > \frac{n^2}{8\pi} \), we suppose to have two different phases. We'll use RG to confirm it.

Let us change the short range cut \( \frac{n^2}{X_\xi} \) and we will have \( \ell \rightarrow \ell' = \ell + \delta \ell \).

See how these coupling constant changes. We separate the fast and slow modes

\[ \Theta_{\delta \ell} = \Theta_{\ell} + \Phi_\ell, \text{ where } \Phi_\ell \text{ is the fast mode, containing modes with wavelengths between } \ell \text{ and } \ell + \delta \ell. \]

\[ (\partial_x \Theta^e) = (\partial_x \Theta^e)^2 + (\partial_x \Phi^e)^2. \] (no mixing after integration \( \int d^3 x \))

\[ \cos n \Theta^e = \cos (n \Theta^e + n \Phi^e) = \cos n \Theta^e \sin n \Phi^e \Phi^e \theta^e + \frac{n^2}{2} \cos n \Theta^e \theta^e \phi^e \]

\[ S = \int d^3 x \left( \frac{X_\xi}{2} (\delta_x \Theta^e)^2 - g \cos n \Theta^e + \frac{X_\xi}{2} (\delta_x \Phi^e)^2 \right) \]

\[ \int d^3 x + n g \cos \Theta^e \Phi^e + \frac{n^2}{2} g \cos \Theta^e \phi^e \]

We will treat \( \Theta^e \) as background field and integrate out the fast field of \( \Phi^e \).
This will generate a number of terms such as $(\Delta x \Delta \psi \epsilon)^2$, $\cos 2n \Delta \psi$, $(\Delta x \Delta \psi)^2$, $\cos 2n \Delta \psi$, ..., but we only look at the term that we've already had.

$$\int d^2 x \: d^2 y \: \frac{n^2 g^2}{\rho} (\Delta x \Delta \psi)^2 \: \langle n \Delta \psi(x) n \Delta \psi(y) \rangle (x-y)^2$$

$$\Rightarrow \quad X_l + \alpha = X_l + \frac{\Delta o}{\rho} x^2 \: K_2 \quad \text{where} \quad K_2 = \int d^2 x \: |x|^2 \: \langle n \Delta \psi(x) n \Delta \psi(y) \rangle$$

$$X_2 = \int \frac{d^2 k}{(2\pi)^2} \: \int d^2 x \: \frac{n^2 x^2}{Xe \: k^2} \: e^{i k \cdot x \cos \theta - \frac{\theta}{\xi} x}$$

$$\frac{2\pi}{l + \alpha} < |h| < \frac{2\pi}{l}$$

$$= \int d^2 x \: \frac{n^2}{Xe} \cdot x^2 \: \int_0^{\frac{2\pi}{l}} d\theta \: e^{i \frac{2\pi}{l} x \cos \theta - \frac{\theta}{\xi} x} \cdot \frac{1}{(2\pi)^2} \ln \left( \frac{l + \alpha}{\xi} \right)$$

$$= \frac{n^2}{Xe} \: \frac{1}{4\pi^2} \: \frac{\alpha}{\xi} \: \int_0^{2\pi} d\theta \: \int_0^{\infty} x^3 dx \: e^{i \frac{2\pi}{l} (\cos \theta + i \frac{\theta}{\xi}) x} \cdot 2\pi$$

$$= \frac{n^2}{Xe} \: \frac{1}{2\pi} \: \frac{\alpha}{\xi} \: \int_0^{\frac{2\pi}{l}} d\theta \: \int_0^{\infty} x^3 dx \: e^{i \frac{2\pi}{l} (\cos \theta + i \frac{\theta}{\xi})} \cdot \int_0^{\infty} x^3 dx' \: e^{i \frac{2\pi}{l} (\cos \theta + i \frac{\theta}{\xi}) x'}$$

$$= \frac{\alpha}{\xi} \: \frac{n^2}{Xe} \: \frac{\ell^4}{32 \pi^5} \: \int_0^{2\pi} d\theta \: \frac{1}{(\cos \theta + i \frac{\theta}{\xi})^4} \cdot 6 = \frac{\alpha}{\xi} \: \frac{3n^2 \ell^4}{16 \pi^5} \int_0^{2\pi} d\theta \: \frac{1}{(\cos \theta + i \frac{\theta}{\xi})^4}$$

$$\Psi \frac{d \chi_e}{d \ln \ell} = \frac{3n^2 \ell^4 (\Delta \psi^2)^2}{16 \pi^4 \chi_e}$$
\[ Z = \int \text{d} \theta \text{d} \phi \text{e}^{- \int \text{d} \theta \left( \frac{k^2}{2} (\text{d}x \cdot \text{d}x) - g \text{e} \text{w} \theta \phi \right) \text{e}^{- \frac{1}{2} \Delta \phi \left( \text{d}x \cdot \text{d}x \right)}} \]

\[ = \int \text{d} \theta \text{d} \phi \text{e}^{\int \text{d} \theta \left( \frac{k^2}{2} (\text{d}x \cdot \text{d}x) - g \text{e} \text{w} \theta \phi \right)} \cdot \exp \left[ \int \text{d} \theta \left( -n \text{e} \text{sin} \theta \phi \text{d} \phi - \frac{n^2}{2} g \text{e} \text{w} \theta \phi \text{d} \phi \right) \right] \]

\[ \Rightarrow \quad \Delta S = \int \text{d} \theta \text{d} \phi \frac{1}{2} g \text{e} \text{w} \theta \phi \text{d} \phi (\theta(0), \theta(0)) \]

\[ - \int \text{d} x \text{d} y \left( \frac{1}{2} (\text{e}^2 \text{sin} \theta \phi(x) \text{sin} \theta \phi(y)) \langle n \text{d} \phi(x) n \text{d} \phi(y) \rangle \right). \]

\[ \langle n^2 \theta \phi \rangle = \frac{\int \text{d} ^2 k}{(2\pi)^2} \text{e}^{\frac{4\pi k^2}{\lambda k x \lambda k}} = \frac{n^2}{2\pi \lambda x} \text{ln} \left( \frac{l + \Delta l}{\Delta l} \right) \]

Thus the first term will change \( g \phi \rightarrow g \phi' = g \phi - \frac{1}{2} g \phi \frac{n^2}{2\pi \lambda x} \text{ln} \left( 1 + \frac{l}{\Delta l} \right) \)

i.e. \( \text{d} g \phi = - g \phi \frac{n^2}{4\pi \lambda x} \frac{\text{d} l}{l} \Rightarrow \frac{\text{d} g \phi}{\text{d} l} = - g \phi \frac{n^2}{4\pi \lambda x} \)

The second term can be represented as

\[ \int \text{d} x \text{d} y \frac{g \phi^2}{4} (\text{sin} \theta \phi(x) - \text{sin} \theta \phi(y))^2 \langle n \text{d} \phi(x) n \text{d} \phi(y) \rangle - \int \text{d} x \text{d} y \frac{g \phi^2}{2} \text{sin} \theta \phi(x) \langle n \text{d} \phi(x) n \text{d} \phi(y) \rangle \]

\[ = \int \text{d} x \text{d} y \frac{g \phi^2}{4} \text{sin} \theta \phi(x) \langle (x \cdot \theta \phi(x))^2 (x - y)^2 \rangle \]

\[ - \int \text{d} x \frac{g \phi^2}{2} \text{sin} \theta \phi(x) \text{d} y \langle n \text{d} \phi(x) n \text{d} \phi(y) \rangle \]
RG flow, fixed point \( \tilde{g} = g l^2 \) and \( \tilde{X} = X \), are dimensionless.

\[
\frac{d \tilde{g}}{d \ln l} = \left( 2 - \frac{n^2}{4 \pi X} \right) \tilde{g} ; \quad \frac{d \tilde{X}}{d \ln l} = \frac{3 n^4 \tilde{g}^4}{16 \pi^4 \tilde{X}}
\]

Let us first ignore the second equation.

\[
\tilde{g}(l) = \tilde{g}(l_0) e^{(2 - \frac{n^2}{4 \pi X}) \ln \left( \frac{l}{l_0} \right)} = \tilde{g}(l_0) \left( \frac{l}{l_0} \right)^{2 - \Delta} \frac{n^2}{8 \pi} \tilde{X}
\]

\( \Delta = \frac{n^2}{4 \pi X} \)

If \( 2 - \Delta > 0 \), no matter how small \( \tilde{g}(l_0) \) is, it can become as large as you want. Let us set the length scale of \( \tilde{g}(l) \sim 1 = \tilde{g}(l_0) \left( \frac{l}{l_0} \right)^{2 - \Delta} \)

\( l = l_0 \left( \frac{1}{\tilde{g}(l_0)} \right)^{2 - \Delta} \), at this scale RG fails, the perturbative method does not apply.

This implies we enter the length scale of correlation length \( \xi \sim l_0 \left( \frac{1}{\tilde{g}(l_0)} \right) \).

( Symmetry breaking state ) \( \langle e^{i \Phi} \rangle \neq 0 \), \( \rightarrow \) vortex condensation.

If \( 2 - \Delta < 0 \), then \( \tilde{g} \) dies exponentially, and we arrive at a symmetric phase.

\( \rightarrow \) power-law superfluid phase.

Near the transition point at \( X = \frac{n^2}{8 \pi} \), the changes of \( \tilde{g} \) and \( \tilde{X} \) are comparable to each other. We need to consider both simultaneously.
Let us linearize the RG around $\bar{x} = \frac{n^2}{8\pi}, \bar{g} = 0$. \[ \frac{d\bar{g}}{d\ln l} = \frac{16\pi}{n^2} \delta\bar{x} \bar{g}, \quad \frac{d\delta\bar{x}}{d\ln l} = \frac{3n^2}{2\pi^3} \bar{g}^2 \]

$\Rightarrow$ \[ \frac{\bar{g} d\bar{g}}{3n^4} - \frac{1}{2} \delta\bar{x} d\delta\bar{x} = 0 \quad \text{i.e.} \quad (\delta\bar{x})^2 = \frac{3n^4}{32\pi^4} \bar{g}^2 + \text{Const} \]

In region I, we have the solution

In region II,

In region III

Regions I, II are approximately described by neglecting the renormalization of $\delta\bar{x}$.

Regions III is the crossover region

we integrate out from $l$ to $\xi$.

At $\xi$, $\bar{g}$ and $\delta\bar{x}$ are

$\Rightarrow$ Relation between $\xi$ and $\delta x(l_0)$.
along the line of

\[ \delta X_c = - \sqrt{\frac{3n^4}{32\pi^4}} \frac{g_c}{3}, \]

\[ \Rightarrow g_{\min} = 0, \]

The left hand side diverges as

\[ \delta X(5) \to 0, \text{ thus } 5 \to +\infty. \]

Let set \( X_{lo} \) slightly larger than \( X_c \), then

\[ -\frac{2}{\delta X_c} = \frac{32\pi^4}{3n^4}. \]

and fix \( g_{lo} \).

\[ \frac{\delta^2}{\delta g_{lo}^2} = \frac{32\pi^4}{3n^4} (\delta X_c)^2 = 0 \]

\[ \frac{\delta^2}{\delta g_{lo}^2} \left[ (\delta X_c)^2 + 2 \delta X_c (X_{lo} - X_c) \right] = g_{\min} \]

\[ \Rightarrow \frac{\delta^2}{\delta g_{\min}^2} = 2 \left( \frac{32\pi^4}{3n^4} \right)^{1/2} \frac{g_{lo}}{g_{\min}} (X_{lo} - X_c) \]

The integration can be performed from \( (\delta X = 0, \ g = g_{\min}) \)

\[ \int_{0}^{\delta X(5)=1} \frac{\delta X}{32\pi^4 (\delta X)^2 + (g_{\min})^2} \approx \int_{0}^{\delta X(5)=1} \frac{d\delta X}{32\pi^4 (\delta X)^2} = \frac{n^2 \ln \left( \frac{5}{\delta_{lo}} \right)}{\sqrt{\frac{3n^4}{32\pi^4} g_{\min}}} \]

\[ \Rightarrow \frac{3n^4}{32\pi^4} \frac{1}{\delta X} \left[ \sqrt{\frac{3}{32}} \frac{n^2}{\pi^2} \frac{1}{g_{\min}} \right] = \frac{n^2 \ln \left( \frac{5}{\delta_{lo}} \right)}{\sqrt{\frac{3n^4}{32\pi^4} g_{\min}}} \]

\[ \Rightarrow 5 \approx lo \sqrt{\frac{n^2}{2\pi} \left( \frac{\delta X}{\delta_{lo} \frac{g_{lo}}{g_{\min}}} \right)} \]

\[ C = \frac{n^2}{2\pi \delta_{lo}^2} \left( \frac{3}{32\pi^2} \right)^{3/2} \]
general principle of RG. Fixed points / phase transitions

Stable fixed points. A, B corresponds to stable phases.

Unstable fixed points controls phase transitions C. The line D→C→D' is the phase boundary between phase A/B.

K-T transition.

Symmetry breaking phase in the clock model.

(low-temperature, superfluid phase

power law - correlation