Lecture 9: Superconductivity — fundamental properties
— de Gennes’ book.

1. Zero resistance below \( T_c \)

2. Diamagnetism: London theory

At \( T < T_c \), we can divide electrons into "normal fraction" and "superconducting fraction". Newton's second law gives to

\[
m \frac{d\vec{J}_s}{dt} = me \frac{d}{dt} (n_s \vec{v}) = me \left( \frac{\partial}{\partial t} (n_s \vec{v}) + \vec{v} \cdot \nabla (n_s \vec{v}) \right) = \nabla \cdot \vec{e} \nabla e \nabla \nabla (n_s \vec{v}) = 0
\]

\[
\frac{\partial n_s}{\partial t} + \nabla \cdot (n_s \vec{v}) = 0
\]

\[
\Rightarrow \quad e \frac{\partial}{\partial t} (n_s \vec{v}) - me \vec{v} \cdot \frac{\partial n_s}{\partial t} = n_s e^2 E \ \Rightarrow \text{for static state}
\]

\[
\frac{\partial J_s}{\partial t} = \frac{n_s e^2 E}{m}
\]

thus

\[
\nabla \times \frac{\partial}{\partial t} J_s = \frac{n_s e^2}{m} \nabla \times E = \frac{n_s e^2}{m} \left[ -\frac{1}{c} \frac{\partial B}{\partial t} \right]
\]

\[
\Rightarrow \quad \frac{\partial}{\partial t} \left( \nabla \times J_s + \frac{n_s e^2}{mc} B \right) = 0
\]

\[
\nabla \times J_s + \frac{n_s e^2}{mc} B = f(r), \text{ which should be time-independent}
\]

London assumed that \( f(r) = 0 \) \Rightarrow \nabla \times \vec{J}_s = -\frac{n_s e^2}{mc} \vec{B}
\[ \nabla \times \vec{J}_s = - \frac{n_s e^2}{mc} \nabla \times \vec{A} \quad \Rightarrow \quad \frac{q}{q} \times \vec{J}_s (q) = - \frac{n_s e^2}{mc} \frac{q}{q} \times \vec{A} (q) \]

* for any finite wave-vector \( \vec{q} \), the susceptibility, (current-current),

\[ \chi_{JJ}^\perp (\vec{q}, 0) = - \frac{n_s e^2}{mc} , \]

but the longitudinal \( \vec{A} \) is a pure gauge, and should not have any response \( \chi_J^{\parallel} (\vec{q}, 0) = 0 \).

We have the superconducting state, \( \lim_{q \to 0} (\chi_{JJ}^{\perp} (q, 0) - \chi_J^{\parallel} (q, 0)) \neq 0 \).

Thus in the long-wave length limit, the system can still distinguish transverse and longitudinal.

\[ \vec{A} \rightarrow \vec{A} \rightarrow \vec{A} \quad \text{transverse} \quad \text{longitudinal} \]

* Penetration depth

\[ \nabla \times (\nabla \times \vec{J}_s) = - \frac{n_s e^2}{mc} \nabla \times \vec{B} = - \frac{4 \pi n_s e^2}{m c^2} \vec{J}_s \]

\[ \nabla (\nabla \cdot \vec{J}_s) - \nabla^2 \vec{J}_s = - \nabla^2 \vec{J}_s = - \frac{4 \pi n_s e^2}{m c^2} \vec{J}_s \]

\[ \Rightarrow \quad \lambda = \frac{4 \pi n_s e^2}{m c^2} \]
Consider if the superconducting state can be described by a macroscopic wave-function

\[ j(r) = -\frac{ie\hbar}{2m^*} (\psi^* r) \left( \nabla^* \frac{ieA(r)}{\hbar c} \right) \psi(r) + c.c. \]

plug in \( \psi(r) = \rho^{1/2}(r) e^{i\phi(r)} \) \( \Rightarrow j_s(r) = \frac{e^*\hbar}{m} \rho(r) \left( \nabla \phi(r) - \frac{e^*}{\hbar c} \right) \)

later on, we will see \( e^* = 2e \), \( m^* = 2m \). If \( \rho(r) = \frac{N_0}{2} \) and const

\[ \Rightarrow \text{Lindem equation.} \]

* flux quantization

Consider we have a multiply-connected geometry.

The flux \( \Phi \) trapped inside the hole.

\[ \oint j_s(r) \, dr = 0 \Rightarrow \oint \nabla \phi(r) \, dr = \frac{e^*}{\hbar c} \oint A \, dr \]

\[ \frac{e^*}{\hbar c} \Phi = 2n\pi \Rightarrow \Phi \text{ quantized in unit of} \, \frac{\hbar 2\pi c}{2e}. \]

\[ \Phi_0 = \frac{hc}{2e} = 2 \times 10^{-7} \text{ G} \cdot \text{cm}^2 \]

3. * type I and type II superconductors

* type I: critical field \( H_c(T) \)
long cylinder: sample in a solenoid

\[ H = \frac{4\pi NI}{cL} \quad N: \text{number of turns} \]

normal state: \[ F_n = \pi r_0^2 L f_n + \pi r_1^2 L \frac{H^2}{8\pi} \]

superconducting state: \[ F_s = \pi r_0^2 L f_s + \pi (r_1^2 - r_0^2) L \frac{H^2}{8\pi} \]

in order to repel the flux, the work done by the induced emf

\[
\int V I dt = \int \left( -\frac{N}{c} \frac{d\phi}{dt} \right) I dt = -\frac{N}{c} (\phi_f - \phi_i) I = \frac{NI}{c} \pi r_0^2 H
\]

\[ = \pi r_0^2 L \frac{H^2}{4\pi} \]

\[ \Rightarrow \text{at } H_c(\tau), \text{ we have equilibrium } \Rightarrow \quad f_n = f_s - \frac{H^2}{8\pi} + \frac{H^2}{4\pi} = f_s + \frac{H^2}{8\pi} \]

another way to derive it is to go through another way to derive it is to go through

the Gibbs free energy \[ G = F - M H \]

at \( H_c \) we have \( G_s(H_c) = G_n(H_c) \)

The normal state \( G \) doesn't depend on \( H \) much \( G_n(H_c) \approx f_n(0) \).

in SC state \( \Delta G = \frac{H dH}{4\pi} \quad (M = -\frac{H}{4\pi}) \)

\[ G_s(H) - G_s(0) = \int \frac{H dH}{4\pi} = \frac{H^2}{8\pi} \]

\[ \Rightarrow G_s(H) = f_s(0) + \frac{H^2}{8\pi} \quad \Rightarrow \quad f_s(0) + \frac{H^2}{8\pi} = f_n(0) \]
Latent heat: \( d \mathcal{G} = -S \mathbf{d} \tau - M \mathbf{d} \mathbf{H} \)

from \( 1 \to 2 \) in normal state

\[-S_n d \tau + M_n d \mathbf{H}\]

from \( 1 \to 2 \) in superconducting state

\[-S_c d \tau + M_s d \mathbf{H}\]

\[\Rightarrow \frac{d H_c}{d \tau} = -\frac{S_n - S_s}{M_n - M_s}\]

\[M_n \approx 0, \quad M_s = (\frac{4\pi}{3})^2 \mathbf{H} \Rightarrow \frac{-d H_c}{d \tau} \cdot \frac{H_c}{4\pi} = S_n - S_s\]

\[C_n - C_s = T \frac{d}{dT} (S_n - S_s) = -T \frac{d}{dT} \left( \frac{d H_c}{d \tau} \cdot \frac{H_c}{4\pi} \right) \text{ at zero field}\]

\[\Rightarrow \lim_{T \to T_c} \left[ C_n - C_s \right] = -T \frac{d}{dT} \left( \frac{d H_c}{d \tau} \right)^2\]

\[S_n - S_s \sim \frac{T}{T_c} \exp \left(-\frac{4\pi}{k_B} \right)\]

- Pippard non-local form (Coherence length).

In the Coulomb gauge, \( \mathbf{j}(\mathbf{r}) = C \cdot \int \frac{(\mathbf{A}(\mathbf{r}) \cdot \mathbf{R}) \mathbf{R} \cdot e^{-R^2/4\sigma} d\mathbf{r}}{R^2} \) \( R \neq 0 \)

where \( \mathbf{R} = \mathbf{r} - \mathbf{r}' \), \( C \) is a constant.

When \( A \) is a slowly-varying variable, we must come back to London equation.
Set \( \mathbf{A} \) along \( z \)-axis \( \Rightarrow \vec{j}(\mathbf{r}) = \vec{A}(\mathbf{r}) \cdot \frac{\alpha s \phi}{R^2} \cdot e^{-\frac{B s^2}{R^2}} \cdot R^2 dR \sin \theta d\phi \)

\[ = C \cdot \frac{s}{3} \cdot \frac{4 \pi}{\lambda} \cdot \vec{A} = -\frac{n_s e^2}{mc} \]

\[ \Rightarrow C = -\frac{3n_s e^2}{4\pi mc s_0} \] \( s_0 \) is the correlation length

Later on, we can microscopically show \( s_0 = \frac{\hbar v_F}{\pi \Delta} \), where \( \Delta \) is the gap in the SC state.

why Pippard proposed this form, is based on the Chambers formula in the normal state:

\[ \vec{j}(\mathbf{r}, \omega) = e^2 \left( \frac{d\mathcal{N}}{d\mathbf{k}} \right) \frac{v_F}{4\pi} \int d\mathbf{r}' \frac{\vec{R} (\mathbf{R} \cdot \vec{E}(\mathbf{r}'))}{R^4} e^{-i \frac{\omega R}{v_F}} \cdot e^{-\frac{R^2}{\lambda^2}}. \]

• Modification of penetration depth in the Pippard limit \( (\xi \gg \lambda_U) \).

London penetration depth applies in the case \( \lambda \gg \xi \), where \( \lambda \) is slow-varying over the length-scale of \( \xi \). For type-I superconductor, actually \( \lambda \ll \xi \), thus \( \lambda \approx \lambda_U = \frac{4\pi n_s e^2}{mc} \) (\( \mathbf{A} \) is not slow-varying at the scale of \( \xi \))

Consider a sample of \( xy \)-plane, \( A \) is only nonzero within a thickness of \( \lambda \), \( \Rightarrow \)

\[ \vec{j}(\mathbf{r}) \propto -\frac{n e^2}{mc} \frac{\lambda}{s_0} \mathbf{A} \] \( (\lambda \ll s_0) \)

self-consistently \( \Rightarrow \)

\[ \frac{1}{\lambda^2} = \frac{4\pi n_s e^2}{mc} \frac{\lambda}{s_0} \Rightarrow \frac{\lambda^3}{s_0} = \lambda^2 \Rightarrow \frac{\lambda}{\lambda_U} = \left( \frac{s_0}{\lambda} \right)^{1/2} > 1 \]
$\xi$ type II - superconductor $H_{c1}$ and $H_{c2}$

Between $H_{c1}$ and $H_{c2}$, vortex state.

Surface energy:

$|\psi|^2 = n_S$

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$H_c = \lambda(0)$

$\xi(T) \rightarrow H_c$

$\xi(T) \rightarrow H_c$

$|\psi|^2 = n_S$

$\xi(T) \rightarrow H_c$

Surface - type II

interface - type I

interface: domain between SC/normal face allows magnetic field to enter, thus reduces diamagnetic energy, but it suppresses superconductivity and cost energy.

for type I - superconductor, interface energy $> 0$, $(\xi \gg \lambda)$.

$H_{c1} = \frac{\Phi_0}{\lambda L^2}$

$H_{c2} = \frac{\Phi_0}{\xi^2}$

\[ H_{c1} = \frac{\Phi_0}{\lambda L^2} \quad \text{and} \quad H_{c2} = \frac{\Phi_0}{\xi^2} \]

$\rightarrow$ form vortex lattice with each vortex carries $\Phi_0$, with a normal core with size of $\xi$. 

$\xi(T) \rightarrow H_c$

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