§1 Fluctuation, dissipation, and linear response

In the lecture, we have shown that if a system with Hamiltonian $H$ subjected to an external perturbation $H_{e}(t) = Be^{-iwt+2t}$, then the physical quantity of operator $A$ at time $t$, satisfies

$$\langle A(t) \rangle = \langle A \rangle - \frac{i}{\hbar} \int_{-\infty}^{+\infty} dt' \ 面 (t-t') \langle[A(t), B(t')]\rangle e^{-iwt'+2t'}$$

where $\langle A \rangle$ is the value of $A$ in thermal equilibrium, $\hat{A}(t)$ and $\hat{B}(t)$ are operators of $A$ and $B$ in the Heisenberg picture of $H$, i.e. $\hat{A}(t) = e^{\frac{i}{\hbar}Ht} A e^{-\frac{i}{\hbar}Ht}$, $\hat{B}(t) = e^{\frac{i}{\hbar}Ht'} B e^{-\frac{i}{\hbar}Ht'}$.

Define the retarded Green function as

$$G_{r}(t-t') = -\frac{i}{\hbar} θ(t-t') \langle[A(t), B(t')]\rangle ,$$

where the $\langle |...| \rangle$ means the expectation value in thermal equilibrium.

1. Lehmann representation:

Prove that: $G_{r}(t-t') = -\frac{i}{\hbar} θ(t-t') \frac{1}{Z} \sum_{mn} \hat{e}^{\beta E_{n}} \langle m | B | n \rangle \langle n | A | m \rangle$

$$\times e^{-\frac{i}{\hbar}(E_{m}-E_{n})t} (e^{\beta(E_{m}-E_{n})}-1), \text{ where } Z = \sum_{m} \hat{e}^{-\beta E_{m}} \text{ and } |m\rangle, |n\rangle$$

are eigenstates of the Hamiltonian $H$. The Fourier transform should be
$$G_r(w) = z^{-1} \sum_{mn} e^{-\beta E_n} \langle m | B | n \rangle \langle n | A | m \rangle \left( \frac{e^{\beta(E_n-E_m)}-1}{\hbar w - (E_n-E_m) + i\eta} \right)$$

We define the spectra function as

$$J(w) = -2 \text{Im} \; G_r(w), \quad \text{and prove that}$$

$$G_r(w) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{J(w')}{w-w'+i\eta} \; dw'.$$

2° Let us set $A = B$, and define the correlation function

$$S(t-t') = \langle A(t) A(t') \rangle = \int_{-\infty}^{+\infty} \frac{dw}{2\pi} e^{iw(t-t')} S(w).$$

Show that

$$S(t-t'=0) = \langle A^2 \rangle = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{J(w)}{e^{\beta w} - 1} \; dw.$$

$\langle A^2 \rangle$ means the ground state fluctuation, and $J(w)$ denotes the dissipation spectra, thus the above expression is called fluctuation - dissipation theorem.

Show that for the case of $A = B$, $J(w) > 0$ at $w > 0$, and $J(-w) = -J(w)$.

3° Consider the example of forced harmonic oscillator

$$H = -\frac{1}{2m} \frac{d^2}{dx^2} + \frac{1}{2} m \omega_0^2 x^2, \quad H_e = -f(t) x(t)$$
define the retarded response function as the displacement-correlation function

\[ \chi(t-t') = -\frac{i}{\hbar} \theta(t-t') \langle [\mathcal{X}(t), \mathcal{X}(t')] \rangle, \]

Calculate \( \chi(\omega) \) and \( J(\omega) \). Show \( \chi(\omega) \) has the pole at \( \pm \omega_0 \).

Show that in the classic limit \( T \to +\infty \),

\[ \langle x^2 \rangle = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{k_0 T}{\omega} J(\omega) d\omega = \frac{k_0 T}{m \omega_0^2} \] as required by the equipartition theorem!

\[ \delta 2: \text{ Sum-rule (f- longitudinal).} \]

Prove that for interacting electron gas, the following relations are exact.

(a) \[ [\mathcal{H}, P_q], P_{-q} ] = - \left( \frac{N q^2}{m} \right) \]

(b) \[ \int_{0}^{+\infty} d\omega \omega \text{ Im } \chi(q, \omega) = + \frac{\Pi}{a} \]

where \( P_q \) is the Fourier component of density operator, \( N \) is number of particle, \( \chi(q, \omega) \) is the density-density response (vacuum polarization), \( \omega_p \): plasmon frequency.
3) Prove that the 3D plasmon dispersion relation is

\[ \omega_p^2 = \omega_p^2 + \frac{3}{5} q^2 v_F^2 \]
where \( v_F \) is the Fermi velocity and \( \omega_p^2 = \frac{4\pi Ne^2}{m} \)
(from the evaluation of zero of \( \varepsilon(q, \omega) \), at \( \frac{\omega}{v_F} \gg 1 \)).

b) Derive the plasma frequency through classical equation of motion.

Hint: use the continuity equation and equation of motion

\[ \frac{\partial n}{\partial t} + \nabla \cdot (n \vec{u}) = 0, \]
\[ m \frac{\partial^2 (n \vec{u})}{\partial t^2} + m \vec{u} \cdot \nabla (n \vec{u}) = -ne \vec{E}. \]

Expand to the leading order fluctuation \( \delta n = n - n_0 \). Show

\[ \frac{\partial^3}{\partial t^3} \delta n = -\frac{4\pi n_0}{m} \delta n \] at the leading order.

4) We consider a Jellium model.

a) Show that if the screened interaction is taken as the Thomas-Fermi form, then in the limit of low density region, the Hartree-Fock energy of an electron with spin up is independent of \( \vec{k} \), or down.

b) Assume a finite polarization \( P = N^+ - N^\downarrow \), find the total Hartree-Fock energy in the limit of a). Express the result as a function of density \( n \), Fermi energy \( \varepsilon_F \), polarization \( P \), and Thomas-Fermi vector \( k_F \).

C) Calculate the spin-susceptibility due to Hartree-Fock correction. Compare it with the free system. Is the system possible to be spin-polarized? (ferromagnetic)