1. In class, we have already learned the Lindhard response function

\[ \chi_0(q, \omega) = \frac{2}{V} \sum_{k} \frac{n_{k+q} - n_k}{\hbar \omega - (E_{k+q} - E_k) + i\eta}, \]

and have evaluated it in 3D systems at the long wave-length limit. Now we consider the situation at 2D.

a) At T = 0 K, and in the limit of \(V_F q \ll \varepsilon_F\), we can express

\[ \chi_0^{2D}(q, \omega) = N(0) f(s), \]

where \(S = \frac{\omega}{V_F q}\) is a dimensionless parameter, and \(N(0)\) is the density of states at the Fermi surface.

Show that

\[ f(s) = \int_0^{2\pi} d\theta \frac{\cos \theta}{s - \omega \cos \theta + i\eta}, \]

where \(\theta\) is the azimuthal angle of momentum \(k\) on the Fermi surface.

b) Evaluate the imaginary part of \(\chi_0^{2D}(q, \omega)\) at \(V_F q \ll \varepsilon_F\), as a function of \(S = \frac{\omega}{V_F q}\). Plot the result. What's the qualitative differences between \(\text{Im} \chi_0(q, \omega)\) at 3D and 2D?
c) The 2D Fourier transform of the Coulomb interaction \( \frac{e^2}{r} \) is \( \frac{2\pi e^2}{q} \). By evaluating the 2D dielectric function (longitudinal)

\[
\varepsilon(q, \omega) = 1 + \frac{2\pi e^2}{q} \chi^2_0(q, \omega),
\]

and finding its zero point, we can find the dispersion relation of 2D plasmons at the longwave length limit. The density response \( \chi^2_0(q, \omega) \) after taking care of the Coulomb interaction at the RPA level is \( \chi^2(q, \omega) = \chi^2_0(q, \omega) / \varepsilon(q, \omega) \).

Prove that the 2D plasmon is gapless, and its dispersion relation is proportional to \( \sqrt{q} \). Finally, the dispersion relation of \( \omega_0(q) \).

d) What is the temperature dependence, as \( T \to 0 \), of the contribution of the 2D plasmon to the specific heat?

e) Consider a bilayer system with two identical planes spaced by a distance \( d \). Find the frequencies of the two branches of plasmon-like modes in the long wave length limit with \( 2d \ll 1 \).

Hint: the Fourier transform of the inter-plane Coulomb potential \( \frac{e^2}{\sqrt{r^2 + d^2}} \) is \( \frac{2\pi e^2}{q} e^{-qd} \).
2. Pomeranchuk instabilities of Fermi liquids

Consider Fermi surface as an elastic membrane in momentum space. Let us disturb it a little bit by creating \( \delta N(\hat{r}) \).

Let us define the angular deformation by integrating out the radius direction

\[
\delta N(\hat{r}) = \int \frac{p^2 dp}{(2\pi)^3} \delta N(p, \nu_p), \text{ where } \nu_p \text{ is the solid angle direction of } \hat{r}.
\]

Then we define the the spherical-harmonic components of \( \delta N(\hat{r}) \) as

\[
\delta N_{\ell m}(\hat{r}) = \sum_{\ell m} \delta N_{\ell m} Y_{\ell m}(\hat{r}).
\]

a) Prove the cost of the kinetic energy due to the Fermi surface deformation is

\[
\frac{\delta E^{(k)}}{V} = 2\pi N(0) \sum_{\ell m} \left( |\delta N_{\ell m}^s|^2 + |\delta N_{\ell m}^a|^2 \right)
\]

where \( \delta N_{\ell m}^s = \delta N_{\ell m} \pm \delta N_{\ell m} \), \( N(0) \) is the density of states at Fermi surface.

b) Prove that the change of the interaction energy is

\[
\frac{\delta E^{(k)}}{V} = 2\pi N(0) \sum_{\ell m} \frac{F_{\ell}}{2\ell + 1} |\delta N_{\ell m}^s|^2 + \frac{F_{\ell}}{2\ell + 1} |\delta N_{\ell m}^a|^2
\]
c) Add two contributions together, and show that Fermi surface will not be stable if \( F_{e}^{S,q} \leq -(2l+1) \), which is called Pomeranchuck instability. Ferromagnetism is one the simplest version of Pomeranchuck instability, which is in the \( F^{q}_{0} \) channel. Draw the Fermi surface shape after Ferromagnetic instability occurs. Can you imagine what will happen if Pomeranchuck instability occurs at channels of \( (l \geq 1) \)?

(The \( F^{q}_{1} \) channel is subtle, and you can neglect it).
3) Spin waves in ferri-magnets.

Ferri-magnets mean a two-sublattice system with different spin-values \( S_A \) and \( S_B \) coupled by anti-ferro magnetic exchange \( J > 0 \). Consider the cubic lattice.

\[
H = J \sum_{i} S_i^A \cdot S_{i+\delta}^B, \quad \text{with} \quad S_i^A \cdot S_i^B > 0, \quad \delta = \pm \hat{x}, \pm \hat{y}, \pm \hat{z}
\]

a) Develop the H-P spin-wave theory for 3D-ferri-magnetic system.

Show that at long wave length limit, there are two branches of spin-wave excitations with dispersions of

\[
\omega_{\mathbf{k}}^\pm = \begin{cases} 
2 \pi J (S_a - S_b) + 4 J a^2 \frac{S_a S_b}{S_a - S_b} k^2, \quad (z = 6 \text{ is the coordination number}) \\
4 J a^2 \frac{S_a S_b}{S_a - S_b} k^2. \quad \text{a is the lattice constant.}
\end{cases}
\]

b) Show that the zero-point fluctuations of \( S_A^2 \) and \( S_B^2 \) are

\[
\langle \Delta S_A^2 \rangle = \langle \Delta S_B^2 \rangle = \frac{1}{2} \sum_{\delta = \pm \hat{x}, \pm \hat{y}, \pm \hat{z}} \left\{ (1 - c^2 \vartheta_{\mathbf{k}}^2)^{-1/2} - 1 \right\} \frac{2}{(2\pi)^2} \frac{d^2 k^2}{(2\pi)^2},
\]

where \( \vartheta_{\mathbf{k}} = a^2 \sum_{\delta = \pm \hat{x}, \pm \hat{y}, \pm \hat{z}} e^{i \mathbf{k} \cdot \delta} \), \( c = \frac{2 (S_a S_b)^{1/2}}{S_a + S_b} \),

\( \Delta S_A^2 = S_A - \langle S_A^2 \rangle \), and \( \Delta S_B^2 = S_B - \langle S_B^2 \rangle \).