Lect 5: Fermi liquid theory (I)

Landau: 1956 $^3$He normal state:

hard core radius $\sim 2.5\AA$, average inter-particle distance $\sim 3.5\AA$

we assume that there are no phase transitions: crystalline order, magnetic order, superfluidity,....

§1 Concept of quasi-particles

Suppose we have a $N$-body ground state $|G\rangle$. At time $t=0$, an extra particle is inserted in the plane wave state $C_k^\dagger$. After a time period of $T$, we check the amplitude of such a particle still in the state of $\Phi_k$ at time $T$.

$$e^{-iHT} C_k^\dagger |G\rangle = \langle G | e^{-iHT} C_k e^{iHT} C_k^\dagger |G\rangle = \langle G | C_k(T) C_k^\dagger(0) |G\rangle$$

we expand $G_k(T)$ in terms of Lehman representation as

$$G_k(T) = \sum \langle G | C_k(T) |m\rangle <m| C_k^\dagger(0) |G\rangle = \sum |\langle G | C_k|m\rangle|^2 e^{-i(Z_m-E_g)T}$$

In many cases, the spectra weight of $|m| C_k^\dagger |G\rangle$ behaves like
\[ | \langle m | G_k | G \rangle |^2 \]

\[ \text{s-function} \]

Continuum
\[ G_k(T) = Z e^{i E_k T} \]
\[ + \sum_n |G_k G_k(n)|^2 e^{-i E_n T} \]

The second part represents a continuum in the energy space, thus leads to a rapid decaying function.

The long time behavior is controlled by the one particle-like excitation.

In the Fermi liquid state \( 0 < z < 1 \). The \( s \)-function spike can be considered as a quasiparticle state.

§2: Adiabatical Continuity:

Each state of the free fermi gas corresponds to a state of the interacting system by tuning on the interaction adiabatically.

At zero temperature, the quasiparticle distribution satisfies

\[ n_{p\sigma}^0 = \begin{cases} 1 & k \leq k_F \\ 0 & k > k_F \end{cases} \]

We can create excitations by moving some quasiparticles inside the fermi sphere into states outside the fermi surface. The quasiparticle energy is defined as

\[ E - E_0 = \sum_{p\sigma} E_{p\sigma} \delta n_{p\sigma}^0 \quad \text{so do} \]

\[ \mathbf{p}' = \sum_{p\sigma} \mathbf{p} \delta n_{p\sigma}^0 \quad \mathbf{s} = \sum_{\sigma} \mathbf{\sigma} \delta n_{p\sigma}^0 \]
Please note that $\hat{P}_{\text{tot}}$ and $\hat{S}_{\text{tot}}$ are conserved quantities through the process of turning interaction.

Fermi gas $\rightarrow$ turn on interaction

Fermi liquid quasi-particle/quasi-hole

bare electron (particle)/hole

Let us expand $\varepsilon(k) = (\frac{d\varepsilon}{dk})_{k=k_F}(k-k_F)$, i.e. $\varepsilon_F = (\frac{d\varepsilon}{dk})_{k=k_F}$

Define effective mass $m^* = \frac{p^2}{\varepsilon_F}$

§ Interactions among quasi-particles

We expand the variation of the ground state energy to the 2nd order of $\delta n_{\rho \sigma}$,

$$\delta E = \sum_{\rho \sigma} \frac{\delta E}{\delta n_{\rho \sigma}} \delta n_{\rho \sigma} + \frac{1}{2} \sum_{\rho \sigma, \rho' \sigma'} \frac{\delta^2 E}{\delta n_{\rho \sigma} \delta n_{\rho' \sigma'}} \delta n_{\rho \sigma} \delta n_{\rho' \sigma'}$$

$$\varepsilon_{\rho \sigma} = \frac{\delta E}{\delta n_{\rho \sigma}}$$

$$\frac{1}{V} f(\vec{p}, \vec{p}'; \sigma \sigma') = \frac{\delta^2 E}{\delta n_{\rho \sigma} \delta n_{\rho' \sigma'}} \rightarrow \text{the unit of } f = \text{energy} \times \text{volume}$$

i.e. Fourier component of interaction

$$\Rightarrow \delta E = \sum_{\rho \sigma} \varepsilon_{\rho \sigma} \delta n_{\rho \sigma} + \frac{1}{V} \sum_{\rho \sigma, \rho' \sigma'} f(\vec{p}, \vec{p}'; \sigma \sigma') \delta n_{\rho \sigma} \delta n_{\rho' \sigma'}$$

$f(\vec{p}, \vec{p}'; \sigma \sigma')$: Landau interaction function
we have

\[ f(pp'; \omega \omega') = f_s(\omega \omega) + f_a(\omega \omega) \omega \omega' \]

where \( f_s = (f_{\uparrow \uparrow} + f_{\uparrow \downarrow})/2 \)

\[ f_a = (f_{\uparrow \uparrow} - f_{\uparrow \downarrow})/2 \]

More generally, spin does not need to be diagonal, and should be represented as density matrix \( \delta \rho_{\alpha \beta}, \omega \beta \), and physical quantities, such as spin, should be represented as \( S = \text{tr}[\hat{S} \delta \rho_{\alpha \beta}] = (\omega \rho_{\alpha \beta} \delta \rho_{\alpha \beta}, \omega \beta \]

the interaction function must generally should be

\[ \frac{1}{2V} \sum_{pp'} \left\{ f_s(\omega \omega') \delta \rho_{\alpha \beta} \delta \omega \omega + f_a(\omega \omega') \delta \rho_{\alpha \beta} \delta \omega \omega \right\} \delta \rho_{\alpha \beta} \delta \omega \omega \]

\( f(pp') \) describes the forward scattering amplitude of quasiparticles near the Fermi surface. \( \omega \omega \)

Symmetry constraint:

- Orbital rotational symmetry \( f(pp') \) can only be a function of \( \hat{p} \cdot \hat{p}' \)

 Spin-rotational symmetry:

\[ \delta \omega \omega \delta \omega \omega \] 

for particle \( p \) and \( p' \)
Calculation of Landau-interaction function at the tree level.

Consider a spin-independent potential, in the 2nd quantization form, we have:

\[ \text{The interaction } H_{\text{int}} = \frac{1}{2} \int d^4x d^4x' \sum_{\alpha, \beta} \psi_\alpha^+ (x) \psi_\alpha^+ (x') V(x-x') \psi_\beta (x') \psi_\beta (x) \]

\[ \text{Fourier transform } = \frac{1}{(2\pi)^4} \sum_{k_1, k_2, q} \psi_\alpha^+ (k_1, q) \psi_\beta (k_2-q) \]

\[ \text{where } V(q) = \int d^4r e^{iqr} V(r) \rightarrow \text{interaction vertex} \]

\[ \begin{array}{c}
\left\langle k_1+q, \alpha \uparrow \right| V(q) \left| k_2-q, \beta \uparrow \rightangle \\
\left\langle k_1 \alpha \downarrow \right| V(q) \left| -k_2 \beta \downarrow \rightangle
\end{array} \]

Fermi liquid interaction function

Corresponds to forward-scattering, i.e. \( q \to 0 \).

However, due to indistinguishable processes, we have

\[ f_{\sigma \rho, \sigma' \rho'} (p, p', \sigma) = V(q=0) \delta_{\sigma \sigma'} \delta_{\rho \rho'} + V(q=0) \delta_{\sigma \sigma'} \delta_{\rho \sigma'} - V(q=0) \delta_{\sigma \sigma'} \delta_{\rho \rho'} \]

\[ \left( \frac{1}{2} \right)^4 \sum_{\sigma} \sum_{\rho} \sum_{\sigma'} \sum_{\rho'} \delta_{\sigma \sigma'} \delta_{\rho \rho'} \delta_{\sigma \sigma'} \delta_{\rho \rho'} \]
using the identity
\[ \frac{1}{2} \left[ \sigma_\alpha \cdot \sigma_\beta + \sigma_\beta \cdot \sigma_\alpha \right] = \delta_\alpha \delta_\beta \]
we have at the tree level
\[ f_{\sigma \tau} (\mathbf{p}, \mathbf{p}') = \left[ V_0 - \frac{1}{2} V_{p-p} \right] \sigma_\alpha \sigma_\beta - \frac{1}{2} V_{p-p} \sigma_\alpha \cdot \sigma_\beta \]

Generally speaking, for system with spin conservation, the Landau interaction function can be represented as
\[ f_{\sigma \tau} (\mathbf{p}, \mathbf{p}') = f^S_{\sigma \tau} (\mathbf{p}, \mathbf{p}') \sigma_\alpha \sigma_\beta + f^A_{\sigma \tau} (\mathbf{p}, \mathbf{p}') \sigma_\alpha \cdot \sigma_\beta. \]

\[ f^S \text{ and } f^A \text{ describes the forward scattering amplitude which marks the fixed points of Fermi liquid in the RG language. } f^S \text{ is in the density channel interacting, while } f^A \text{ is in the spin channel.} \]