Solution to HW 3

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**Problem 1** (Griffiths 5.24)

If \( B \) is uniform, show that \( A(r) = -\frac{1}{2}(r \times B) \) works. That is, check that \( \nabla \cdot A = 0 \) and \( \nabla \times A = B \). Is this result unique, or are there other functions with the same divergence and curl?

**Solution:** Since \( B \) is uniform, \( \nabla \times B = 0 \), \((r \cdot \nabla) B = 0\). And \( \nabla \times r = 0 \), \( \nabla \cdot r = 3 \), we have

\[
\nabla \cdot A = -\frac{1}{2} \nabla \cdot (r \times B) = -\frac{1}{2} (B \cdot (\nabla \times r) - r \cdot (\nabla \times B)) = 0
\]

\[
\nabla \times A = -\frac{1}{2} \nabla \times (r \times B) = -\frac{1}{2} (r (\nabla \cdot B) + (B \cdot \nabla) r - B (\nabla \cdot r) - (r \cdot \nabla) B)
\]

\[
= -\frac{1}{2} (0 + B - 3B - 0) = B.
\]

Take \( A' = A + \nabla \varphi \),

\[
\nabla \cdot A' = \nabla \cdot A + \nabla^2 \varphi,
\]

\[
\nabla \times A' = \nabla \times A.
\]

So we need \( \varphi \) to be linear in \( x, y \) and \( z \) so that \( \nabla^2 \varphi = (\partial_x^2 + \partial_y^2 + \partial_z^2) \varphi = 0 \). For example, take \( \varphi = xy \), \( \nabla \varphi = ye_x + xe_y \), \( \nabla^2 \varphi = 0 \).

**Problem 2** (Griffiths 5.29)

Use the results of Ex. 5.11 to find the field inside a uniformly charged sphere of total charge \( Q \) and radius \( R \), which is rotating at a constant angular velocity \( \omega \).

**Solution:** In Ex. 5.11, we found the vector potential inside a uniformed charged shell with radius \( R' \) as Eq. 5.67,

\[
A(r, \theta, \phi) = \begin{cases} 
\mu_0 \frac{R' \omega}{r} \sin \theta \hat{\phi}, & (r \leq R) \\
\mu_0 \frac{R' \omega}{r} \frac{1}{r^2} \sin \theta \hat{\phi}, & (r \geq R)
\end{cases}
\]

Here, a uniformly charged sphere can be thought as layers of spheres, larger one containing smaller ones inside. The field inside a uniformly charged sphere can be found by integration over \( R' \),

\[
A(r, \theta, \phi) = \frac{\mu_0 \omega}{3} r \sin \theta \hat{\phi} \int_r^R R'dR' + \frac{\mu_0 \omega}{3} \frac{1}{r^2} \sin \theta \hat{\phi} \int_0^R R'^4 dR'
\]

\[
= \frac{\mu_0 \omega}{3} r \sin \theta \hat{\phi} \left( \frac{1}{2} R^2 - \frac{1}{2} r^2 \right) + \frac{\mu_0 \omega}{3} \frac{1}{r^2} \sin \theta \hat{\phi} \left( \frac{1}{5} R^5 - \frac{1}{5} r^5 \right)
\]

In 3D spherical coordinates, the metric is

\[
\eta = \begin{pmatrix} h_r & 0 & 0 \\
0 & h_\theta & 0 \\
0 & 0 & h_\phi \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\
0 & r & 0 \\
0 & 0 & r \sin \theta \end{pmatrix}
\]
\[ \mathbf{B}(r, \theta, \varphi) = \nabla \times \mathbf{A}(r, \theta, \varphi) \]

\[ = \frac{1}{h_\varphi} \left[ \frac{\partial}{\partial \theta} (A_\varphi h_\varphi) - \frac{\partial}{\partial \varphi} (A_\theta h_\varphi) \right] \hat{f} + \frac{1}{h_\varphi h_r} \left[ \frac{\partial}{\partial \varphi} (A_\varphi h_r) - \frac{\partial}{\partial r} (A_\varphi h_\varphi) \right] \hat{\theta} + \frac{1}{h_r h_\theta} \left[ \frac{\partial}{\partial \theta} (A_r h_\theta) - \frac{\partial}{\partial r} (A_\theta h_\theta) \right] \hat{\phi} \]

\[ = \frac{1}{r^2 \sin \theta} \left[ \frac{\partial}{\partial \theta} (A_r r \sin \theta) \right] \hat{r} + \frac{1}{r \sin \theta} \left[ - \frac{\partial}{\partial r} (A_r r \sin \theta) \right] \hat{\theta} \]

\[ = \frac{\mu_0 \omega}{2} \frac{Q}{3 \pi R^3} \left[ \frac{\sin \theta}{\sin \theta} \left( \frac{1}{3} R^2 - \frac{r^2}{5} \right) \right] \hat{r} - \frac{1}{r \sin \theta} \frac{\partial}{\partial r} \left( \frac{1}{3} R^2 r^2 - \frac{r^4}{5} \right) \hat{\theta} \]

\[ = \frac{\mu_0 Q}{4\pi R} \left[ \cos \theta \left( 1 - \frac{3}{5} \frac{r^2}{R^2} \right) \hat{r} - \sin \theta \left( 1 - \frac{6 r^2}{5 R^2} \right) \hat{\theta} \right]. \]

**Problem 3 (Griffiths 5.30)**

(a) Complete the proof of Theorem 2, Sect. 1.6.2. That is, show that any divergenceless vector field \( \mathbf{F} \) can be written as the curl of a vector potential \( \mathbf{A} \). What you have to do is find \( A_x, A_y \) and \( A_z \) such that: (i) \( \partial A_x / \partial y - \partial A_y / \partial z = F_z \); (ii) \( \partial A_x / \partial z - \partial A_z / \partial x = F_y \); and (iii) \( \partial A_y / \partial x - \partial A_x / \partial y = F_x \). Here’s one way to do it: Pick \( A_x = 0 \), and solve (ii) and (iii) for \( A_y \) and \( A_z \). Note that the “constants of integration” here are themselves functions of \( y \) and \( z \)—they’re constant only with respect to \( x \). Now plug these expressions into (i), and use the fact that \( \nabla \cdot \mathbf{F} = 0 \) to obtain

\[ A_y = \int_0^x F_z(x', y, z) \, dx'; A_z = \int_0^y F_x(0, y', z) \, dy' - \int_0^z F_y(x', y, z) \, dx'. \]

**Solution:** Pick \( A_x = 0 \),

\[ -\partial A_z / \partial x = F_y \Rightarrow A_z = -\int_0^x F_y(x', y, z) \, dx' + C_1(y, z), \]

\[ \partial A_y / \partial x = F_z, \Rightarrow A_y = \int_0^x F_z(x', y, z) \, dx' + C_2(y, z). \]

Now plug these expressions into (i),

\[ \frac{\partial}{\partial y} \left[ -\int_0^x F_y(x', y, z) \, dx' + C_1(y, z) \right] - \frac{\partial}{\partial z} \left[ \int_0^x F_z(x', y, z) \, dx' + C_2(y, z) \right] = F_x, \]

\[ -\int_0^x \left( \frac{\partial}{\partial y} F_y(x', y, z) + \frac{\partial}{\partial z} F_z(x', y, z) \right) \, dx' + \frac{\partial}{\partial y} C_1(y, z) - \frac{\partial}{\partial z} C_2(y, z) = F_x, \]

and use the fact that \( \nabla \cdot \mathbf{F} = 0 \) to get

\[ \int_0^x \frac{\partial}{\partial x} F_x(x', y, z) \, dx' + \frac{\partial}{\partial y} C_1(y, z) - \frac{\partial}{\partial z} C_2(y, z) = F_x, \]

\[ \Rightarrow \frac{\partial}{\partial y} C_1(y, z) - \frac{\partial}{\partial z} C_2(y, z) = F_x(0, y, z). \]

Take \( C_2(y, z) = 0 \),

\[ A_y = \int_0^x F_z(x', y, z) \, dx', \]

\[ C_1(y, z) = \int_0^y F_x(0, y', z) \, dy', \]

\[ A_z = -\int_0^x F_y(x', y, z) \, dx' + C_1(y, z) \]

\[ = -\int_0^x F_y(x', y, z) \, dx' + \int_0^y F_x(0, y', z) \, dy'. \]
Prob. 5.51. (b) By direct differentiation, check that the $A$ you obtained in part (a) satisfies $\nabla \times A = F$. Is $A$ divergenceless? This was a very asymmetrical construction, and it would be surprising if it were—although we know that there exists a vector whose curl is $F$ and whose divergence is zero.

Solution:

\[
\nabla \times A = \begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
0 & f_0^x F_z(x', y, z) \, dx' - f_0^y F_y(x', y, z) \, dx' + f_0^y F_x(0, y', z) \, dy' \\
\end{vmatrix}
\]

\[
= i \left( - \int_{0}^{x} \frac{\partial}{\partial y} F_y(x', y, z) \, dx' + \frac{\partial}{\partial y} \int_{0}^{y} F_z(0, y', z) \, dy' - \int_{0}^{x} \frac{\partial}{\partial z} F_z(x', y, z) \, dx' \right)
\]

\[
- j \frac{\partial}{\partial x} \left( - \int_{0}^{x} F_y(x', y, z) \, dx' + \int_{0}^{y} F_x(0, y', z) \, dy' \right) + k \frac{\partial}{\partial x} \int_{0}^{x} F_z(x', y, z) \, dx'
\]

\[
= i \left( - \int_{0}^{x} \frac{\partial}{\partial y} F_y(x', y, z) + \frac{\partial}{\partial z} F_z(x', y, z) \right) \, dx' + F_x(0, y, z)
\]

\[
+ j \frac{\partial}{\partial x} \int_{0}^{x} F_y(x', y, z) \, dx' + k \frac{\partial}{\partial x} \int_{0}^{x} F_z(x', y, z) \, dx'
\]

\[
= i \left( \int_{0}^{x} \frac{\partial}{\partial y} F_y(x', y, z) \, dx' + F_x(0, y, z) \right) + \frac{\partial}{\partial x} \left( \int_{0}^{x} F_y(x', y, z) \, dx' \right) + k \frac{\partial}{\partial x} \int_{0}^{x} F_z(x', y, z) \, dx'
\]

\[
= i F_x(x, y, z) + j F_y(x, y, z) + k F_z(x, y, z) = F
\]

\[
\nabla \cdot A = \int_{0}^{x} \frac{\partial}{\partial y} F_y(x', y, z) \, dx' - \int_{0}^{x} \frac{\partial}{\partial z} F_z(x', y, z) \, dx' + \int_{0}^{y} \frac{\partial}{\partial y} F_y(0, y', z) \, dy'
\]

\[
\ne 0,
\]

in general.

(c) As an example, let $F = y \hat{x} + z \hat{y} + x \hat{z}$. Calculate $A$, and confirm that $\nabla \times A = F$. (For further discussion see Prob. 5.51.)

Solution: Let $F = y \hat{x} + z \hat{y} + x \hat{z}$.

\[
A_y = \int_{0}^{x} F_z(x', y, z) \, dx' = \int_{0}^{x} x' \, dx' = \frac{1}{2} x^2,
\]

\[
A_z = - \int_{0}^{x} z \, dx' + \int_{0}^{y} y' \, dy' = -xz + \frac{1}{2} y^2.
\]

\[
A = \left( \frac{1}{2} x^2 \hat{y} + \frac{1}{2} y^2 - xz \right) \hat{z},
\]

\[
\nabla \times A = \left( \frac{\partial}{\partial y} A_z - \frac{\partial}{\partial z} A_y \right) \hat{x} + \left( - \frac{\partial}{\partial x} A_z \right) \hat{y} + \left( \frac{\partial}{\partial x} A_y \right) \hat{z}
\]

\[
= y \hat{x} + z \hat{y} + x \hat{z}.
\]

Problem 4 (Griffiths 5.36)

Find the magnetic dipole moment of the spinning spherical shell in Ex. 5.11. Show that for points $r > R$ the potential is that of a perfect dipole.

Solution:

\[
m = \int dm = \int I dA = \int \frac{dq}{dt} dA = \hat{z} \int_{0}^{\pi} \sigma \left( \frac{2\pi R \sin \theta}{R} \right) R \sin \theta \, d\theta = \pi \sigma R^4 \omega \hat{z}.
\]
For points $r > R$ the potential is

$$A(r, \theta, \varphi)|_{r>R} = \mu_0 R^4 \omega \sigma \frac{1}{3} \frac{1}{r^2} \sin \theta \hat{\phi}. $$

$$A_{dp} = \mu_0 \frac{m \times \hat{r}}{4\pi \frac{r^2}{r^2}} = \mu_0 \frac{4\pi \sigma R^4 \omega}{4\pi \frac{3}{3}} \hat{z} \times \hat{r} = \mu_0 R^2 \omega \sigma \frac{1}{3} \frac{1}{r^2} \sin \theta \hat{\phi} = A(r, \theta, \varphi)|_{r>R}. $$