

# Slater and Mott Physics of $SU(N)$ Hubbard models

Congjun Wu

Department of Physics, University of California, San Diego

1. D. Wang, Lei Wang, CW, arXiv:1907.01748.
2. Shenglong Xu, Julio Barreiro, Yu Wang, CW, Phys. Rev. Lett. 121, 167205 (2018)
3. D. Wang, Y. Li, Z. Cai, Z. Zhou, Yu Wang, CW, Phys. Rev. Lett. 112, 156403 (2014).
4. Z. C. Zhou, CW , Yu Wang, Phys. Rev. B 97, 195122 (2018).
5. Z. C. Zhou, D. Wang, Zi Yang Meng, Yu Wang, CW , Phys. Rev. B 93, 245157 (2016).
6. Hsiang-hsuan Hung, Yupeng Wang, CW, Phys. Rev. B 84, 054406 (2011).

## Collaborators

Da Wang

(UCSD → Nanjing Univ.)

Yi Li

(UCSD → Princeton → Johns Hopkins)

Zi Cai

(UCSD → Innsbruck → Shanghai JT)

Hsiang-hsuan Hung

(UCSD → UIUC → UT Austin → Industry)

Yu Wang, ZhiChao Zhou

(Wuhan Univ.)

Collaborators on earlier works: S. C. Zhang (Stanford),  
J. P. Hu, S. Chen and Y. P. Wang (IOP, CAS).

Supported by NSF, AFOSR.



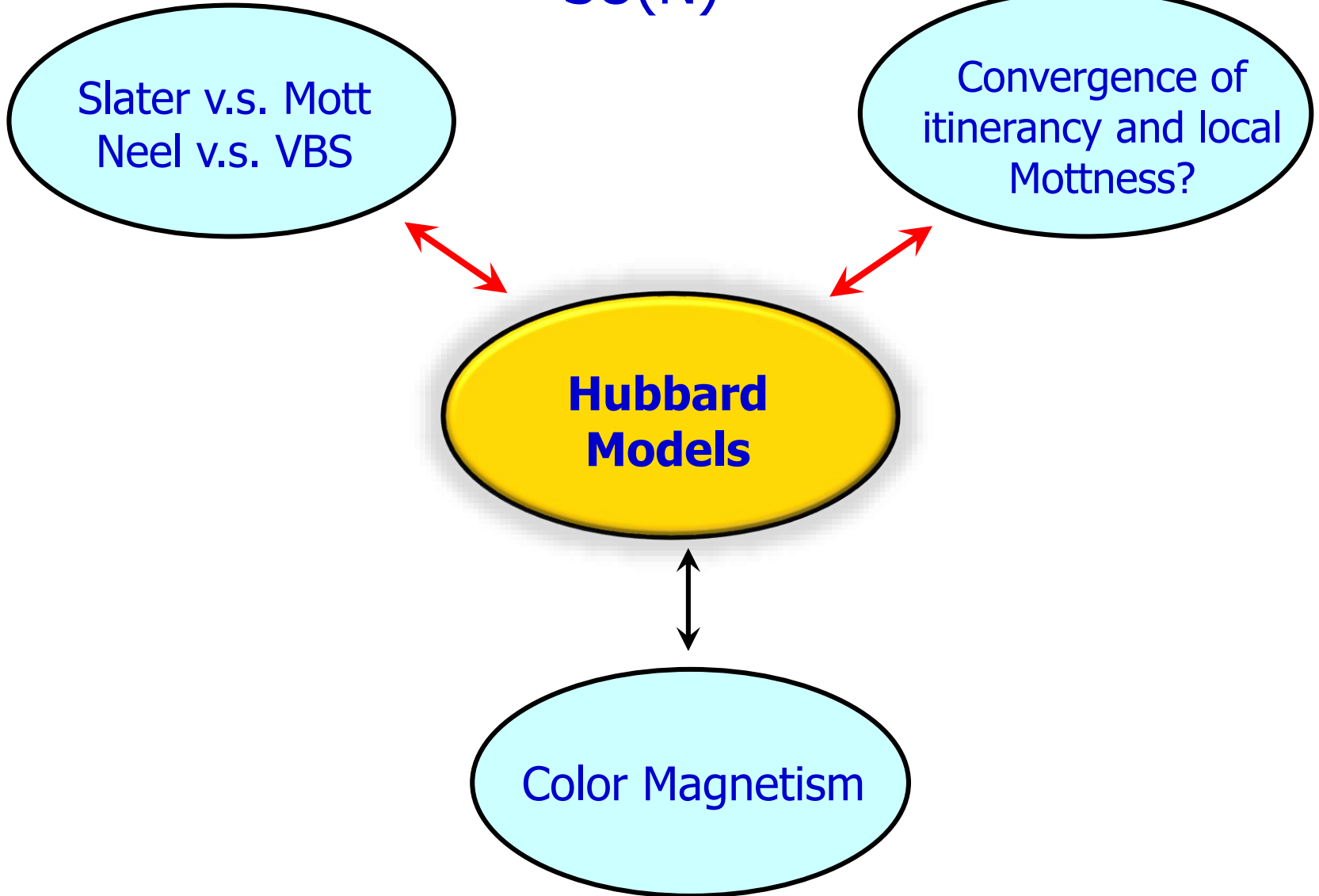
SU(N)

Slater v.s. Mott  
Neel v.s. VBS

Convergence of  
itinerancy and local  
Motttness?

**Hubbard  
Models**

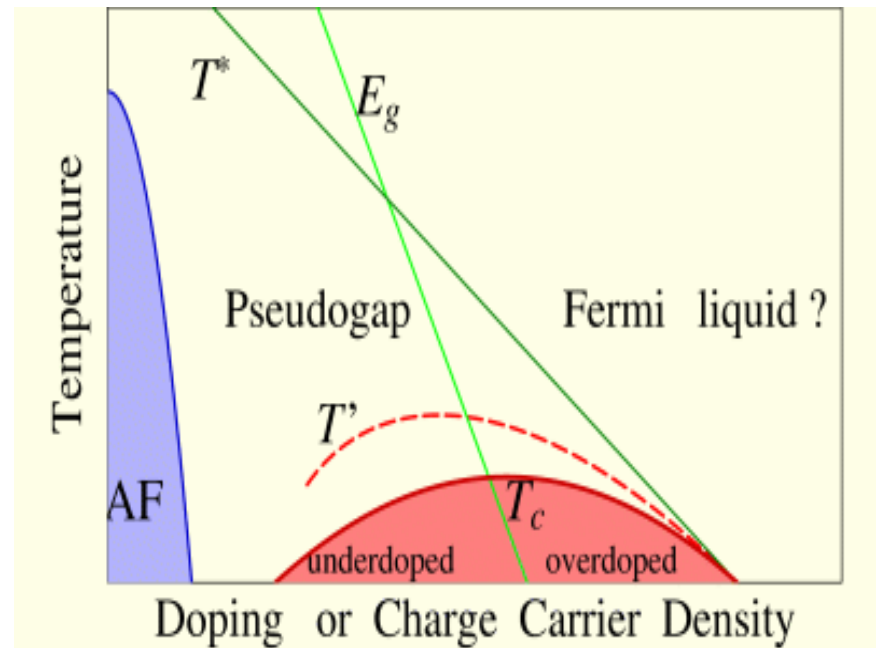
Color Magnetism



# The Hubbard model

$$H = -t \sum_{\langle ij \rangle, \sigma} c_{i\sigma}^+ c_{j\sigma} + h.c. - \mu \sum_{i, \sigma} c_{i\sigma}^+ c_{i\sigma} + U \sum_{i, \sigma} n_{i\uparrow} n_{i\downarrow}$$

- Hubbard 1963: originally for itinerant ferromagnetism (FM) --- unsuccessful.
- Successful for metal-Mott insulator transitions.
- High  $T_c$  cuprates?  
--- Still in debates.



# 1D Mott state – Absence of long-range order

- Half-filled ( $U>0$ ) – Lieb-Wu solution (Hubbard).

1. Charge gap opens at  $U \rightarrow 0$ : Relevance of the Umklapp term
2. Spin channel is critical  $\rightarrow$  no symmetry breaking
3. Bosonization+Sine-Gordon, DMRG

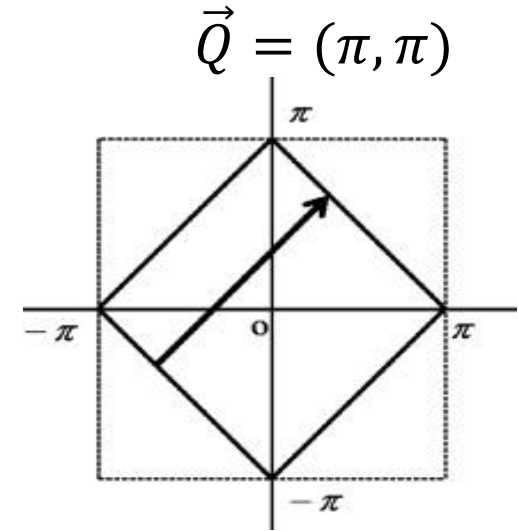
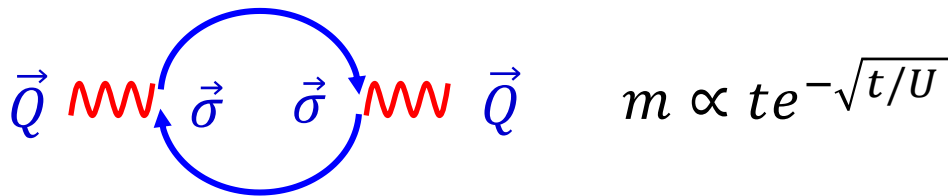
- Bethe ansatz solution for the Heisenberg model.

1. Ground state energy  $E/N = \frac{1}{4} - \ln 2$
2. Fractionalized gapless excitation: spinon

C. N. Yang, PRL 19, 1312 (1967); Lieb and F. Y. Wu, PRL 20, 1445, (1968).

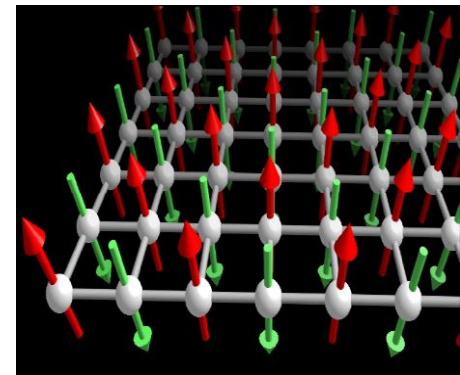
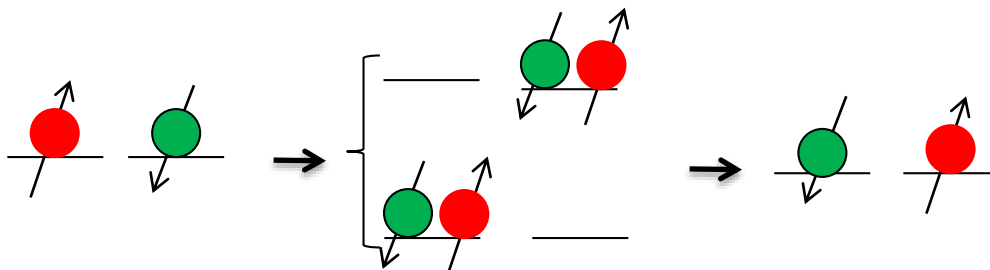
# 2D: Slater v.s. Mott (half-filling)

- Fermi surface nesting ( $U/t \rightarrow 0$ ) : strong charge fluctuations.



- Local moments ( $U/t \rightarrow \infty$ ) : AF super-exchange.

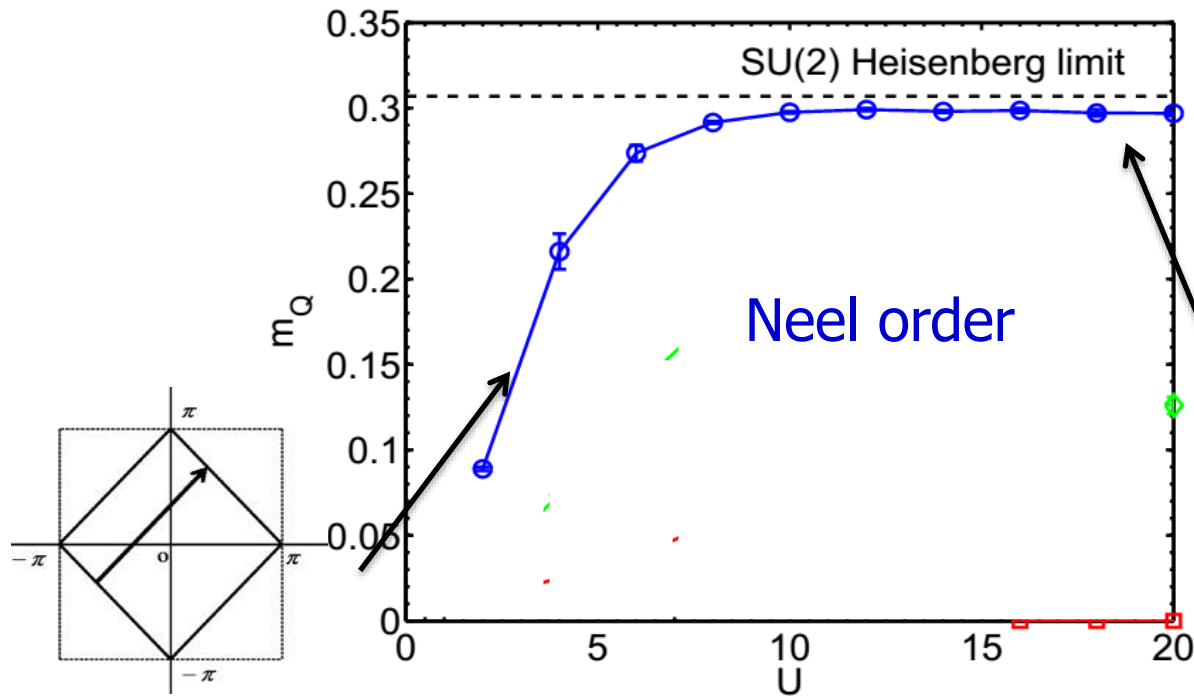
$$H = J \sum \vec{S}_i \cdot \vec{S}_j$$



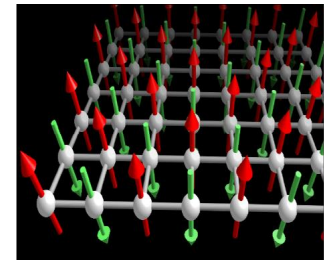
# Crossover: Slater $\rightarrow$ Mott

- AF long-range order for all values of  $U$ .

Determinant QMC: J. Hirsch, 1985.



D. Wang, Y. Li, Z. Cai, Z. Zhou, Y. Wang, **CW**, Phys. Rev. Lett. 112, 156403 (2014).



Blackenbecker, Scalapinio, Sugar, PRD (1981); J. Hirsch, PRB 31, 4403 (1985).

## SO(4) symmetry – pseudo-spin SU(2)

- Yang and Zhang's  $\eta$ -pairing: the charge channel.

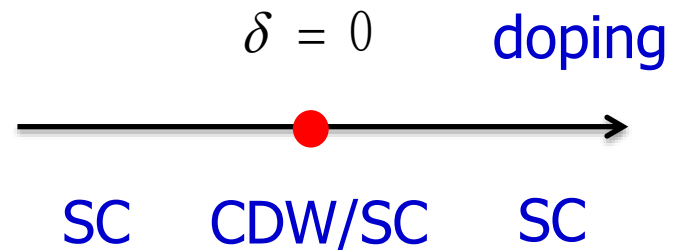
$$\eta^- = \sum_i (-)^i c_{i\downarrow} c_{i\uparrow}, \quad \eta^+ = \sum_i (-)^i c_{i\uparrow}^+ c_{i\downarrow}^+, \quad [\eta^-, \eta^+] = 2N,$$

- Degeneracy between charge-density-wave (CDW) and superconductivity (SC) at half-filling ( $U < 0$ )

$$O_{cdw} = \sum_i (-)^i n_i, \quad \Delta = \sum_i c_{i\uparrow} c_{i\downarrow}, \quad \Delta^+ = \sum_i c_{i\downarrow}^+ c_{i\uparrow}^+, \quad [\eta^+, \Delta] = O_{cdw},$$

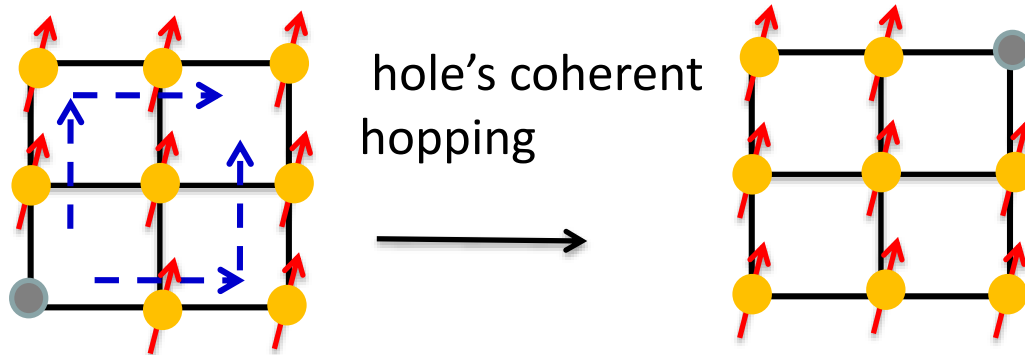
- Pseudo-Goldstone:  $\eta$ -mode

$$H(\eta^+ | G_{sc} \rangle) = (\mu - \mu_0) \eta^+ | G_{sc} \rangle,$$





# Nagaoka FM: infinite-U + single hole



$$H = -t \sum_{\langle ij \rangle, \sigma} P \left\{ c_{i\sigma}^\dagger c_{j\sigma} + c_{j\sigma}^\dagger c_{i\sigma} \right\} P$$

P: no double occupancy

- Ground state WF positive definite – Perron-Frobenius

Bases:  $|\psi(h_j, \{\sigma_i\})\rangle = (-)^j c_{\sigma_1}^\dagger(i_1) \dots c_{\sigma_2}^\dagger(j-1) c_{\sigma_2}^\dagger(j+1) \dots |vac\rangle$ .

- Nearly arbitrary graph (e.g. honeycomb, diamond) -- 15 puzzle problem.



E. Bobrow, K. Stubis, Yi Li, Phys. Rev. B 98, 180101 (2018)

## Phase string – hole's motion

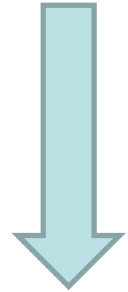
- The Marshall sign can be absorbed.

$$H = -t \sum_{\langle ij \rangle, \sigma} P \left\{ c_{i\sigma}^\dagger c_{j\sigma} + c_{j\sigma}^\dagger c_{i\sigma} \right\} P$$

$$+ J \sum_{\langle ij \rangle} \vec{S}_i \cdot \vec{S}_j$$

- The hole exchanges with a spin  $\downarrow \rightarrow$  a minus sign.

$$c_{i\sigma} \rightarrow c_{i\sigma}, c_{j\sigma} \rightarrow (-)^\sigma c_{j\sigma}$$



- Hole's motion frustrates the WF.

$$H = t \sum_{\langle ij \rangle, \sigma} P \left\{ -c_{i\uparrow}^\dagger c_{j\uparrow} + c_{i\downarrow}^\dagger c_{j\downarrow} + h.c. \right\} P$$

$$+ J \sum_{\langle ij \rangle} \{ -\vec{S}_x \cdot \vec{S}_x - \vec{S}_y \cdot \vec{S}_y + \vec{S}_z \cdot \vec{S}_z \}$$

# Transition metal oxides (large S → classical)

- **Large spin magnitude** from Hund's coupling.
- Inter-site coupling: exchange **a single pair** of electrons.

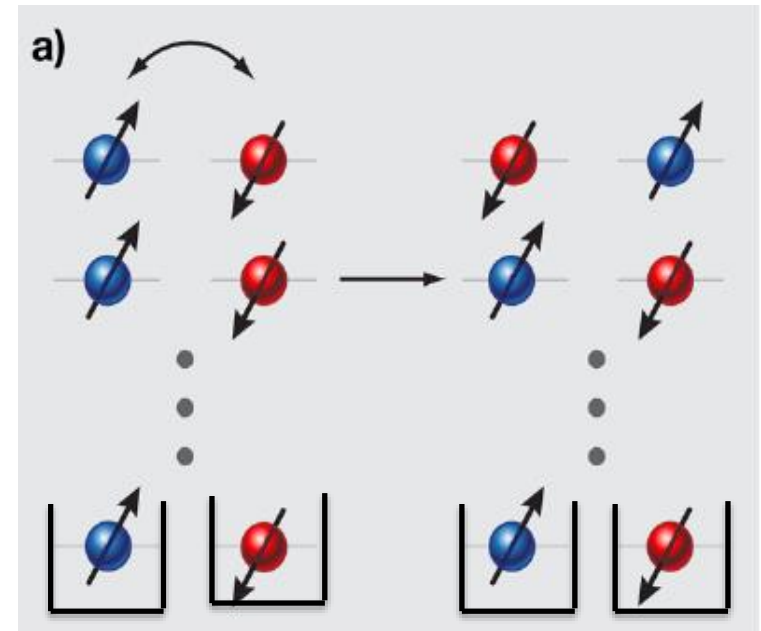
- **1/S-fluctuations:  $\Delta S_z = \pm 1$**

- Bilinear exchange dominates

$$\frac{t^2}{U} \vec{S}_i \cdot \vec{S}_j + \frac{t^4}{U^3} (\vec{S}_i \cdot \vec{S}_j)^2 + \dots$$

C. Wu, Physics 3, 92 (2010).

C. Wu, Nature Physics 8, 784 (2012) (News and Views).



# Large-spin cold fermions inaccessible in solids

- **A new view point: high symmetries,  $Sp(2N)$ ,  $SU(2N)$ .**

$Sp(4)$ ,  $SO(5)$ ,  $SU(4)$  : (**spin  $-\frac{3}{2}$** )  $^{132}\text{Cs}$ ,  $^9\text{Be}$ ,  $^{135}\text{Ba}$ ,  $^{137}\text{Ba}$ ,  $^{201}\text{Hg}$

C. Wu, S. C. Zhang, S. Chen, Y. P. Wang, A. Tsvelik, G. M. Zhang, Lu Yu, X. W. Guan, Azaria, Lecheminant, et al. (2003 ---).

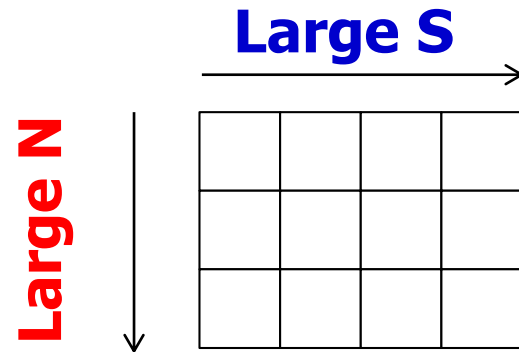
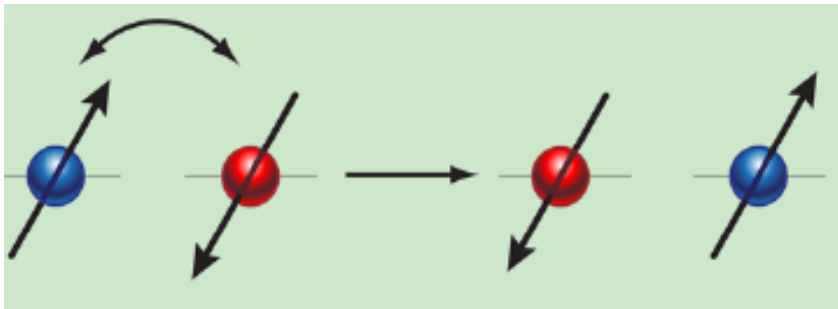
$SU(2N)$ : V. Guriare, M. Hermele, A. Rey, E. Demler, M. Lukin, P. Zoller, et al. (2010 ---).

- What is large? --- **High symmetry** ( $SU(2N)$ ,  $Sp(2N)$ ) rather than **large spin magnitude** (large  $S$ ).

# Cold fermions: large $N \rightarrow$ enhanced fluctuations!

- One step of super-exchange can completely overturn spin configurations.

$$\Delta S_z = \pm 1, \pm 2, \dots \pm S$$



- Bilinear, bi-quadratic, bi-cubic terms, etc.,  $\rightarrow$  equal importance.

$$\vec{S}_i \cdot \vec{S}_j, (\vec{S}_i \cdot \vec{S}_j)^2, (\vec{S}_i \cdot \vec{S}_j)^3$$

# Spin-3/2 Hubbard model

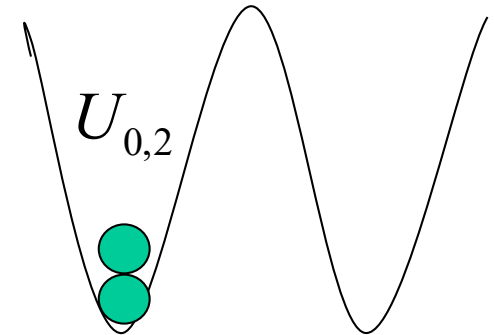
$$H = \sum_{\langle ij \rangle, \alpha} -t \{c_{i,\alpha}^+ c_{j,\alpha} + h.c.\} - \mu \sum_i c_{i,\alpha}^+ c_{i,\alpha} + U_0 \sum_i \eta^+(i) \eta(i) + U_2 \sum_{a=1 \sim 5} \chi_a^+(i) \chi_a(i)$$

$$\begin{array}{cc} \uparrow & \left| \frac{3}{2} \right\rangle \\ \uparrow & \left| \frac{1}{2} \right\rangle \\ \downarrow & \left| -\frac{1}{2} \right\rangle \\ \downarrow & \left| -\frac{3}{2} \right\rangle \end{array}$$

- Fermi statistics: only  $F_{\text{tot}}=0, 2$  are allowed.

singlet:  $\eta^+(i) = \sum_{\alpha\beta} \langle 00 | \frac{3}{2} \frac{3}{2}; \alpha\beta \rangle c_{\alpha}^+(i) c_{\beta}^+(i)$

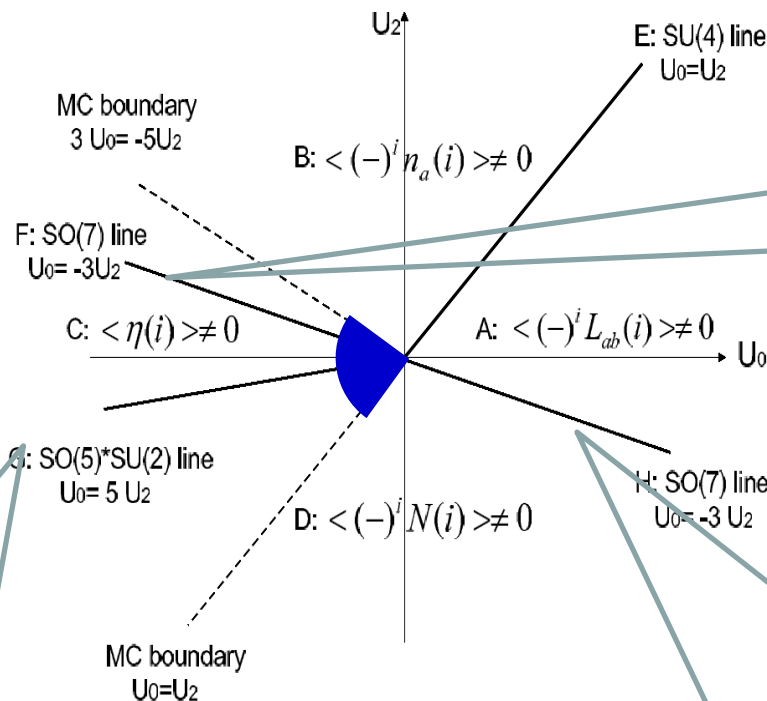
quintet:  $\chi_a^+(i) = \sum_{\alpha\beta} \langle 2a | \frac{3}{2} \frac{3}{2}; \alpha\beta \rangle c_{\alpha}^+(i) c_{\beta}^+(i)$



- **Exact**  $Sp(4)$ , or,  $SO(5)$  symmetry: arbitrary values of  $t$ ,  $\mu$ ,  $U_0$ ,  $U_2$  and lattice geometry.

# Grand unification under symmetry principle

- $4^2 = 1 + 3 + 5 + 7$ : charge, spin, quadrupole, octupole  
 spin (3) + spin octupole (7)  $\rightarrow$  10:  $Sp(4)$  or  $SO(5)$



SO(7) vector  
 7 = singlet SC (2) + AF  
 spin quadrupole (5)

CW, J. P. Hu, S. C. Zhang, Phys. Rev. Lett. 91, 186402(2003).

[https://wucj.physics.ucsd.edu/teach/Phy239\\_2019/phy239.html](https://wucj.physics.ucsd.edu/teach/Phy239_2019/phy239.html)

Pseudospin SO(3) vector  
 7 = singlet SC (2) + CDW  
 (1)

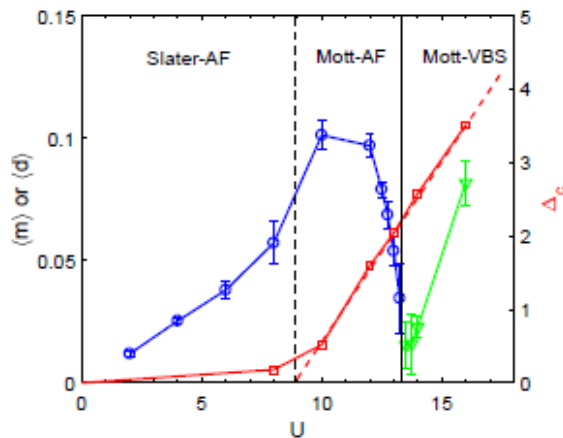
"Grand unification" via SO(7) – adjoint  
 21 = CDW (1) + quintet SC (10) + AF  
 spin (3) and octupole (7)

Slater v.s Mott  
Neel v.s VBS

Convergence of  
itinerancy and local  
Mottness?

**SU(N)**  
**Mott Physics**

Color Magnetism

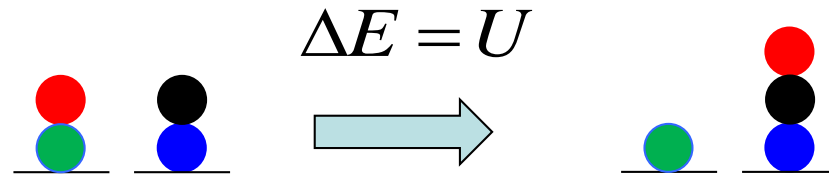




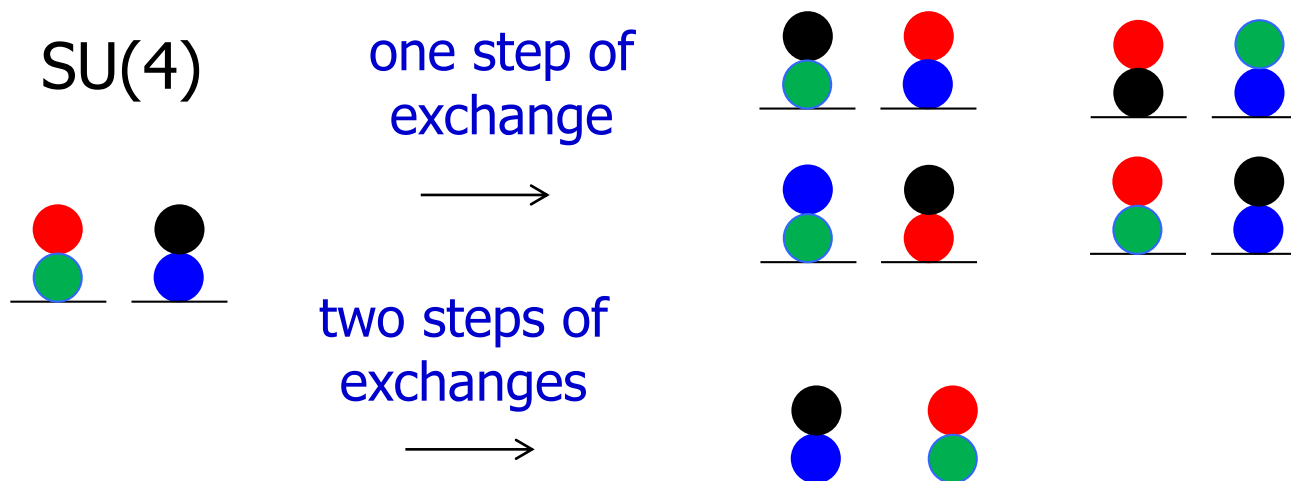
# Half-filled SU(2N) Hubbard model

$$H = -t \sum_{\langle ij \rangle, \sigma=1}^N \{ c_{i\sigma}^+ c_{j\sigma} + h.c. \} + \frac{U}{2} \sum_{i,\sigma} (n_i - N)^2$$

- Atomic limit  $t=0$ .



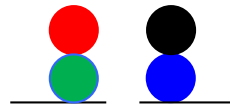
- # of super-exchange processes scales as  $N^2$ .



# Enhanced spin fluctuations

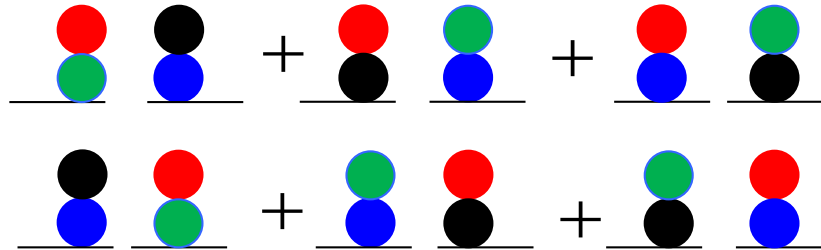
- Neel state unfavorable as N increases.

$$\Delta E = -2zN \frac{t^2}{U}$$



classic-Neel

$$\Delta E = -2zN(N+1) \frac{t^2}{U}$$

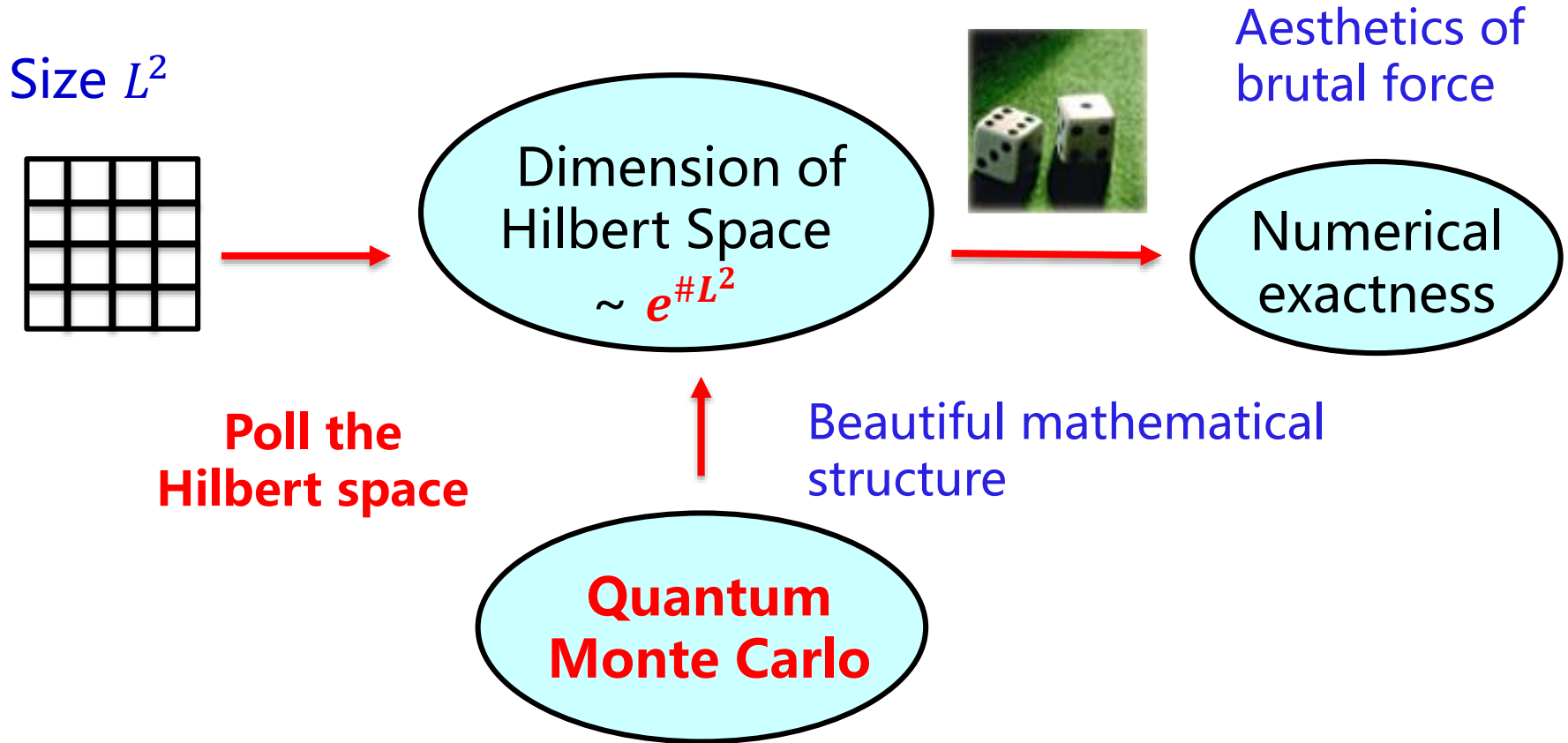


SU(2M)  
singlet

- Dimer state consists of  $\binom{2N}{N}$  resonating configurations.

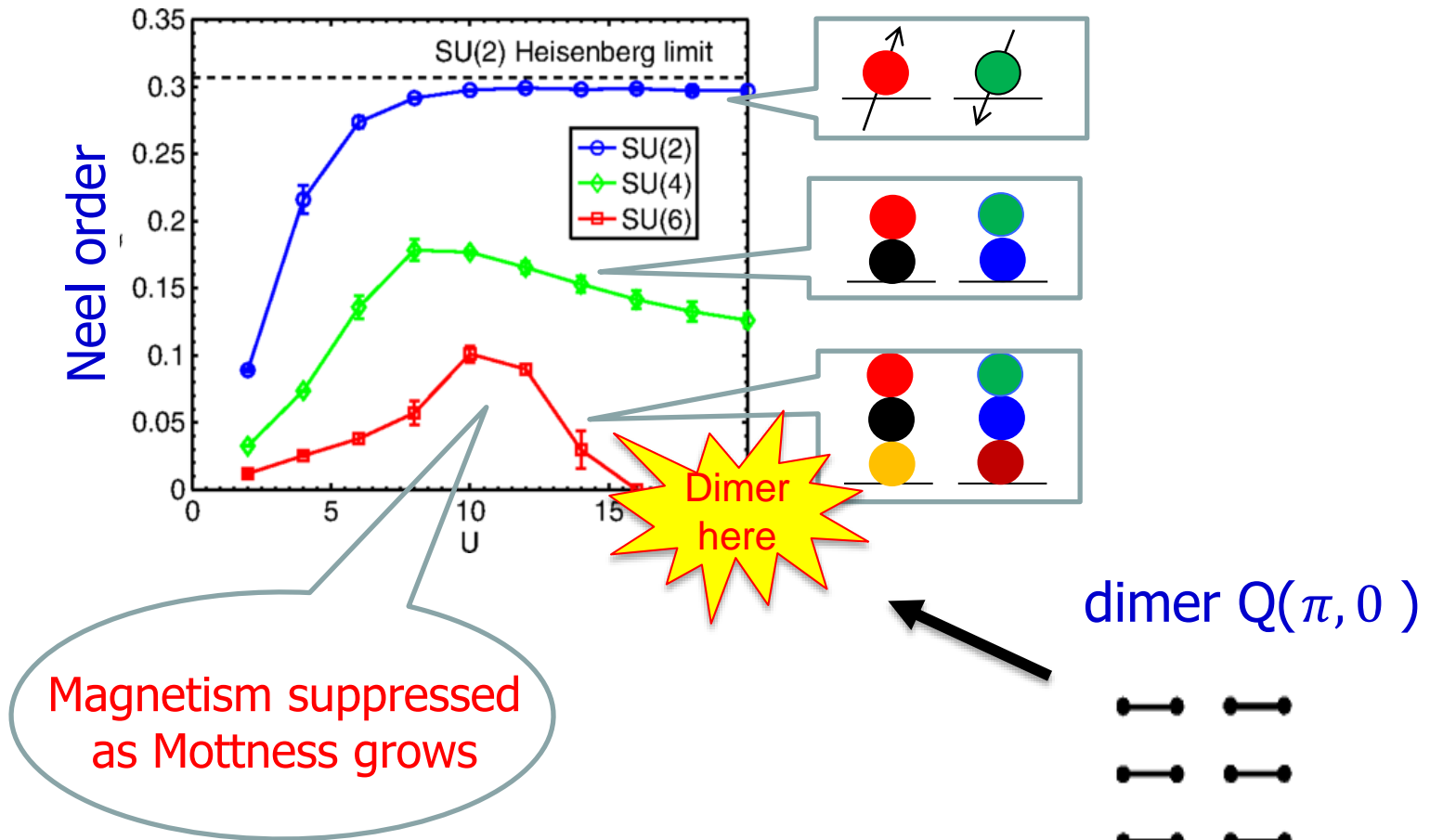
- As  $N > z$  (coordination number), valence bond dimerization favorable (Sachdev + Read).

# Quantum Monte Carlo Simulations



- Condensed matter (Magnetism and Superconductivity), high energy, nuclear physics

# AFM: non-monotonic dependence on U



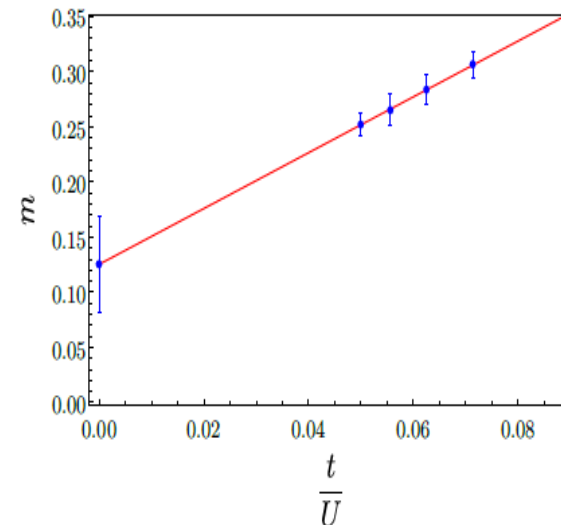
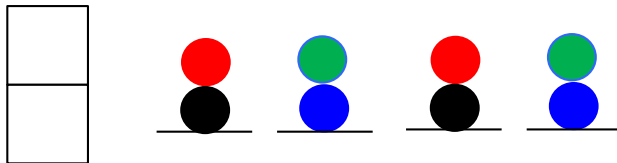
# SU(4): Heisenberg model (self-conjugate)

- Stronger quantum fluctuations: AF long-range order with a smaller moment

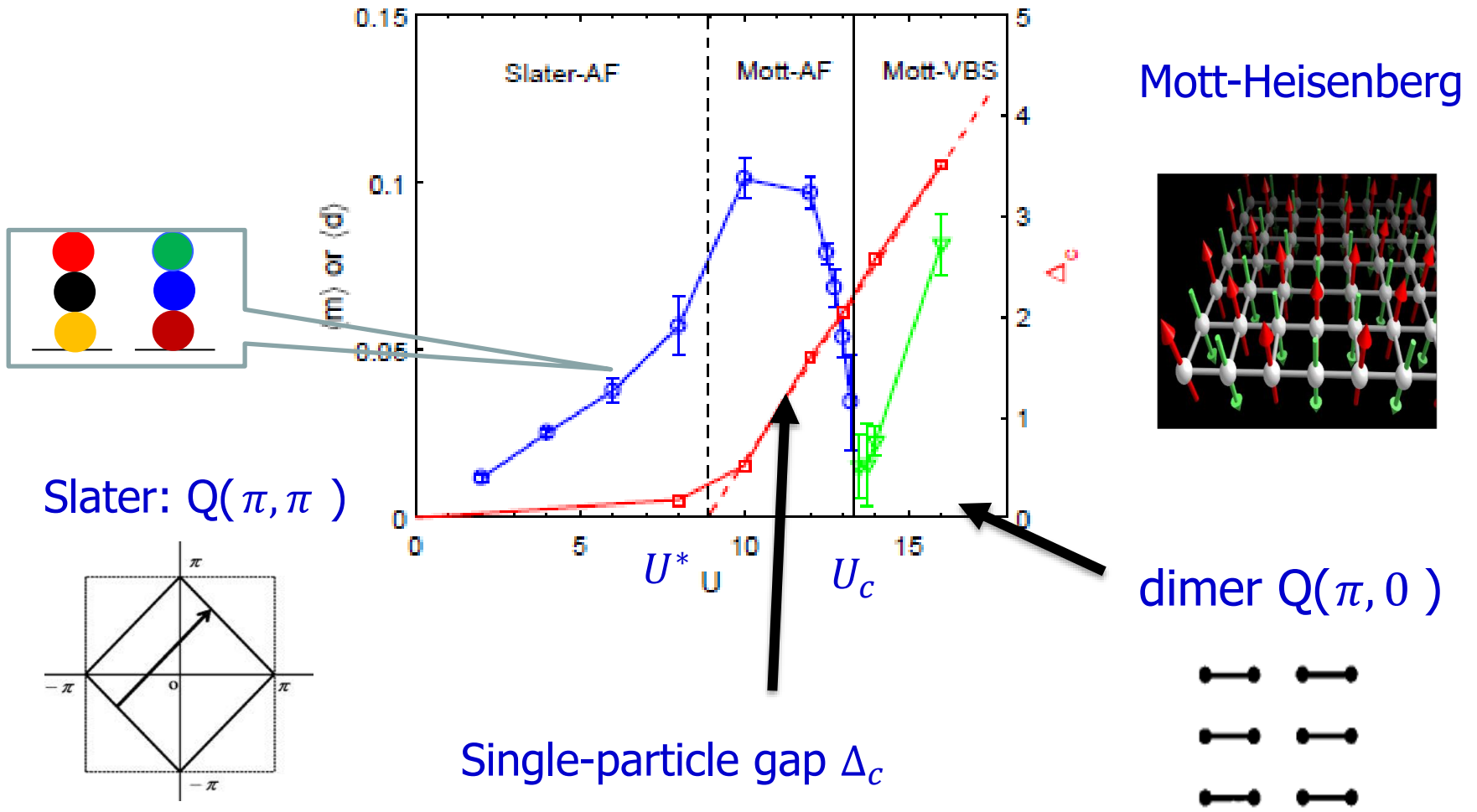
$$U \rightarrow \infty: m = \frac{1}{4} \langle (-)^i (n_1 + n_2 - n_3 - n_4) \rangle \approx 0.11 \pm 0.04$$

Assaad et al, arXiv 1906.06938

- Consistent with  $1/U$  scaling of  $m$  based on our data.

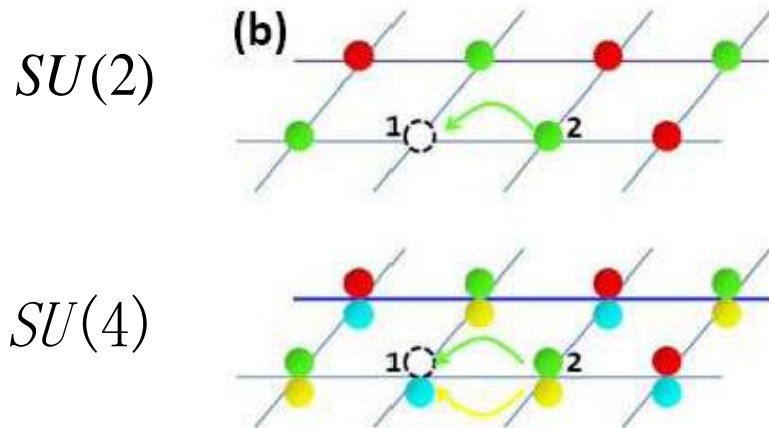
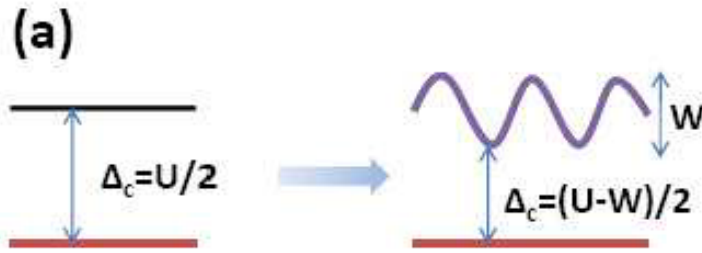


# SU(6): Slater $\rightarrow$ Mott, quantum phase transitions

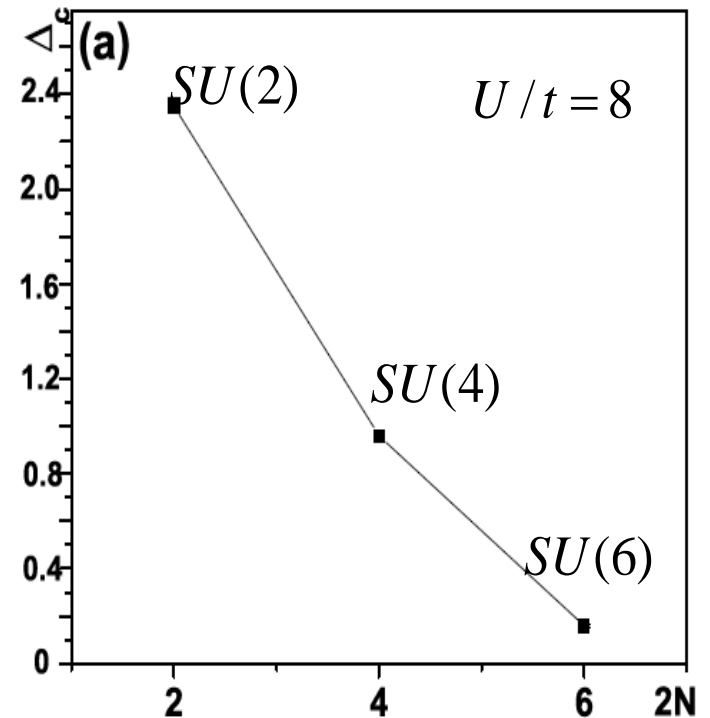


# Single-particle gap softened as N increases

- Hole's band width  $W \sim Nt$ .



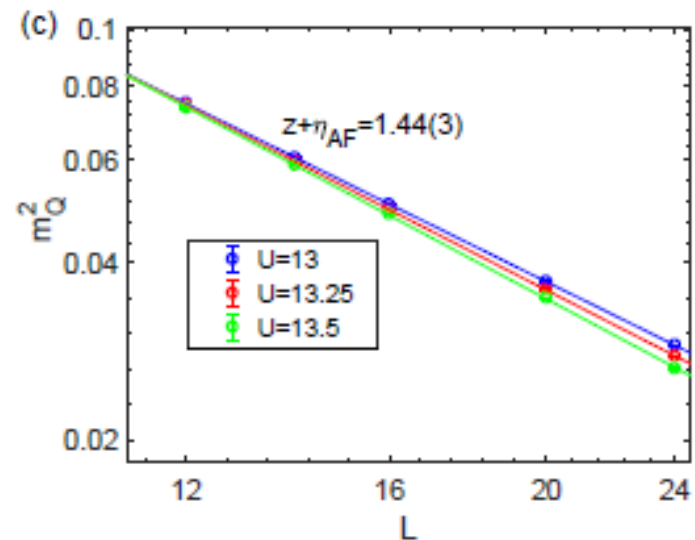
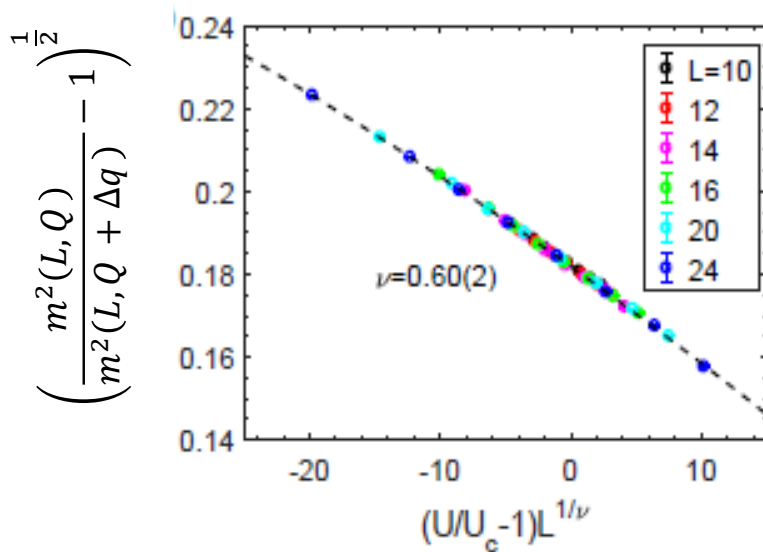
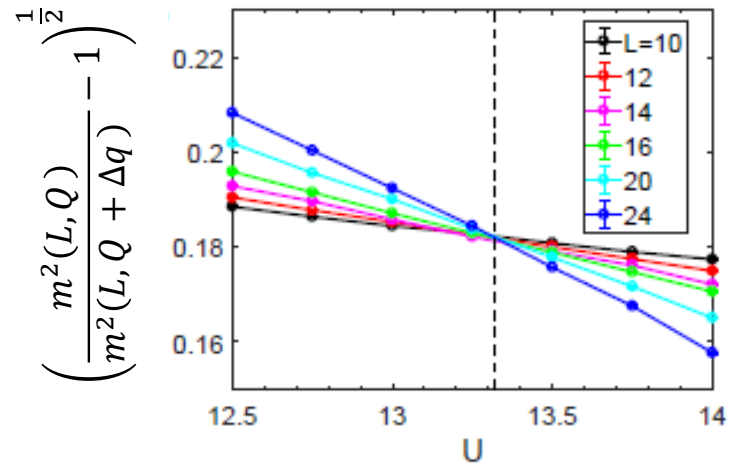
$$\Delta_{sg} = U - W \sim U - Nt$$



# Critical exponents of the AF transition

- A continuous transition.

- $U_c = 13.3 \pm 0.05$ ,
- $\nu = 0.6 \pm 0.02$
- $\eta = 0.44 \pm 0.03$





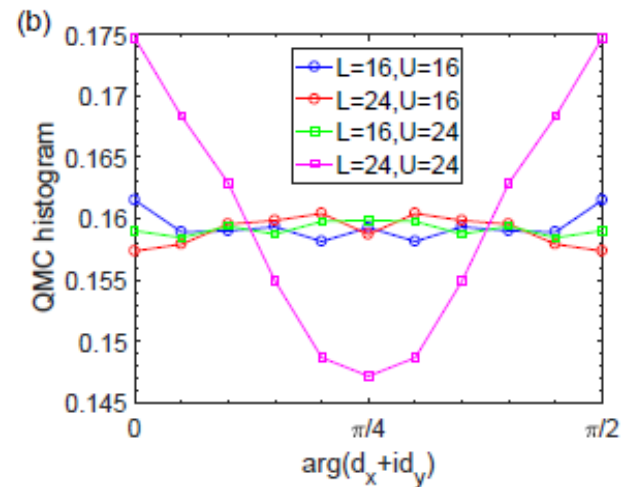
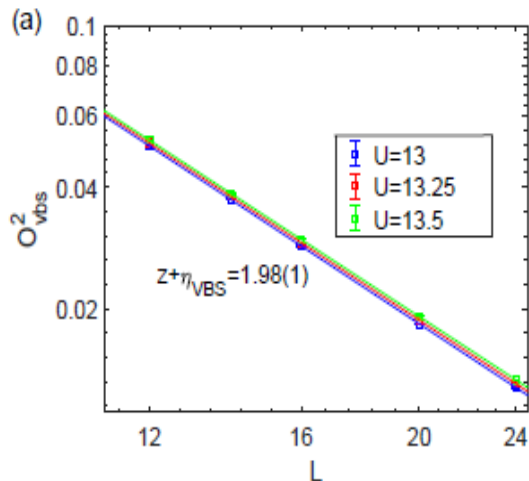
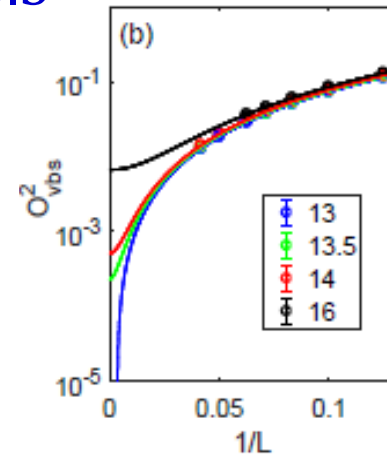
# Dimerization -- VBS

- Structure factor fitting:  $13 < U_c < 13.5$

$$d_{i,\hat{e}_j} = \frac{1}{N} \sum c_{i\alpha}^+ c_{i+e_j,\alpha} + h.c.$$

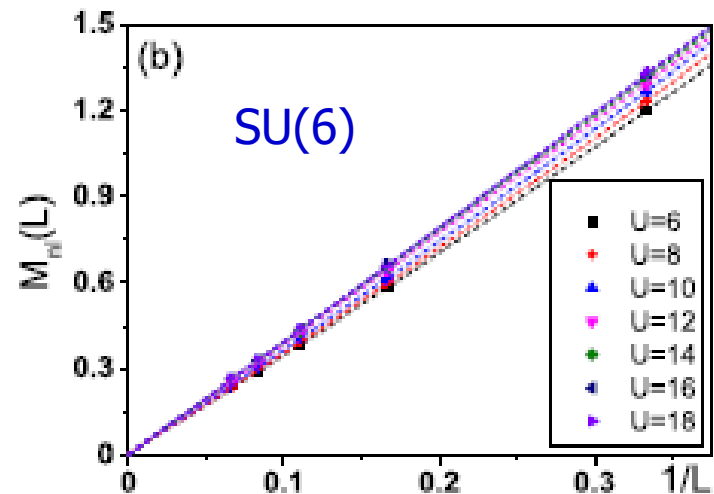
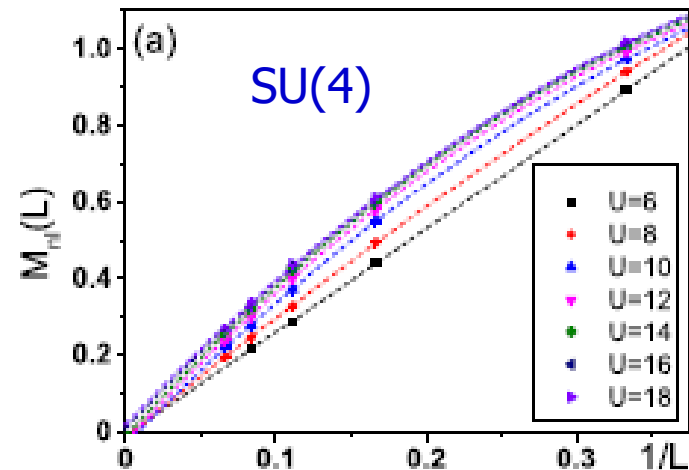
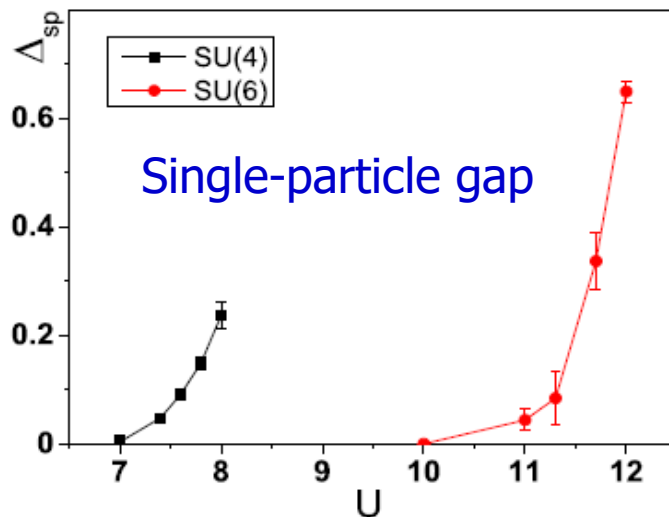
- Columnar VBS when deep inside the VBS phase.

- Data insufficient to judge deconfined criticality or not



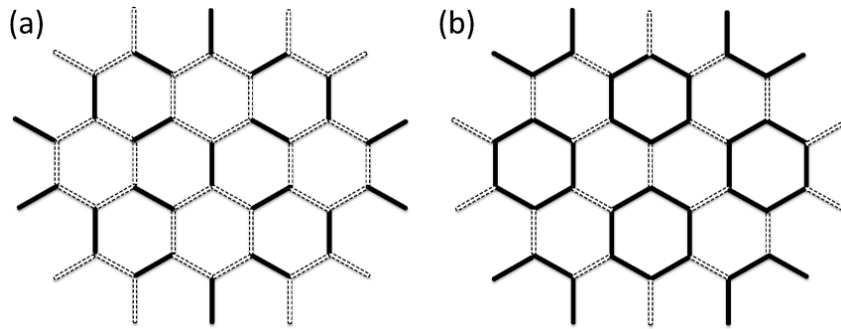
# Mott transition of Dirac-fermion (honeycomb)

- AF  $\rightarrow$  0 for all values of  $U$ .
- Mottness starts later than the case of square lattice:  $U^* \approx 11$  for SU(6).



# Dirac semi-metal $\rightarrow$ VBS transition

$$\psi = \frac{1}{L^2} \sum (d_{i,e_a} + \omega d_{i,e_b} + \omega^2 d_{i,e_c}) e^{i\vec{K} \cdot \vec{r}_i}$$



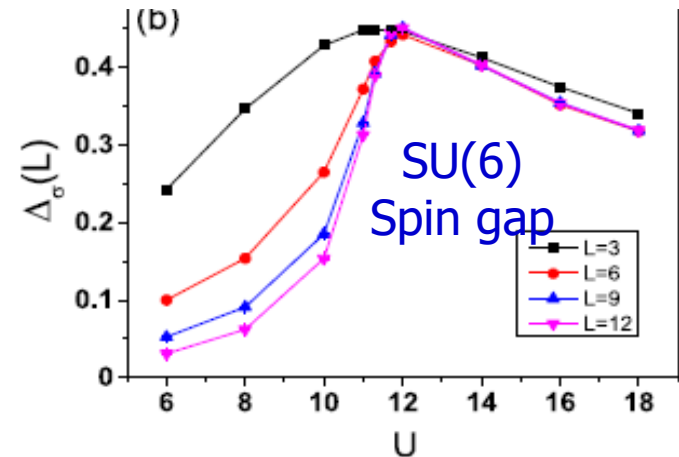
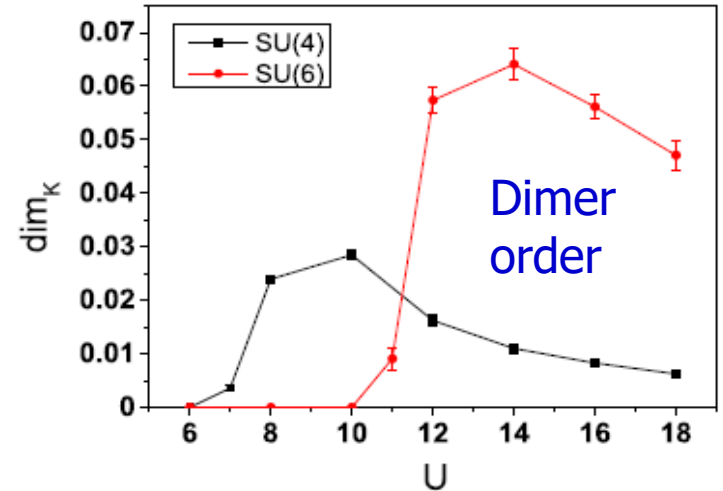
cVBS:

$$\arg \psi = 0, \frac{2\pi}{3}, \frac{4\pi}{3}$$

pVBS:

$$\arg \psi = \pi, \frac{\pi}{3}, \frac{5\pi}{3}$$

$$f_A(\psi, \psi^*) = r_2 |\psi|^2 - r_3 (\psi^{*3} + \psi^3) + r_4 |\psi|^4.$$



Z. C. Zhou, D. Wang, Zi Yang Meng, Yu Wang, **CW**, Phys. Rev. B 93, 245157 (2016).

Hong Yao et al, Nature Communications 8, 314 (2017)

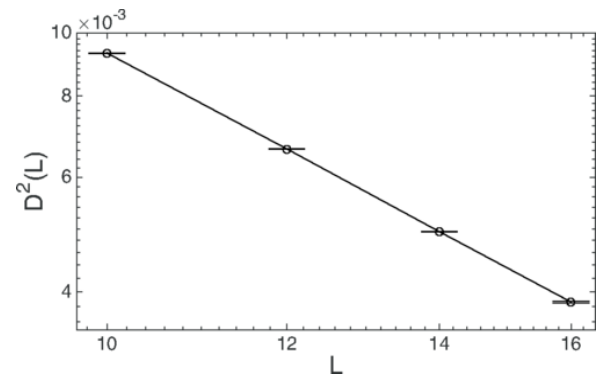
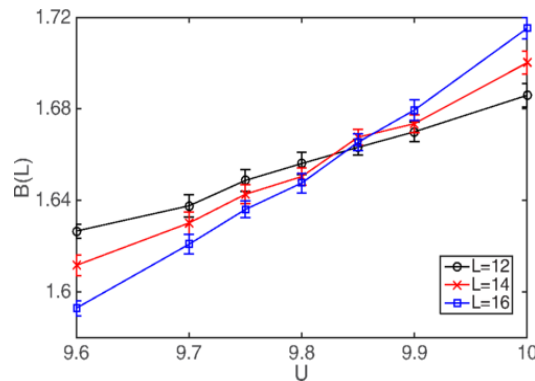
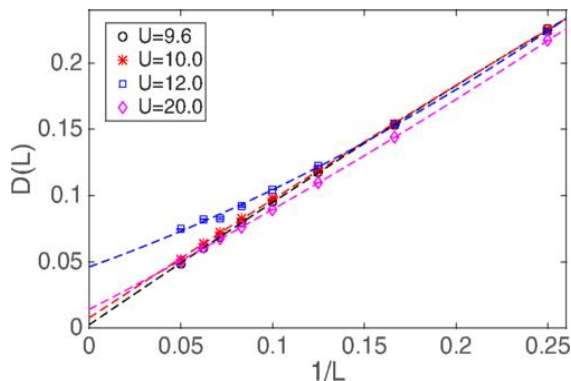
# SU(4) Hubbard model ( $\pi$ -flux square lattice)

- Absence of the AF order at until  $U \leq 20$ , but expect to appear at  $U \rightarrow \infty$
- VBS order appears at  $U_c \approx 9.8 \sim 9.9$  via Binder ratio scaling. Non-monotonic dependence on  $U$ .

$$B_{1,2}^{x,y} = \chi_{x,y}(L, Q_{x,y}) / \chi_{x,y}(L, Q_{x,y} + \Delta q_{1,2})$$

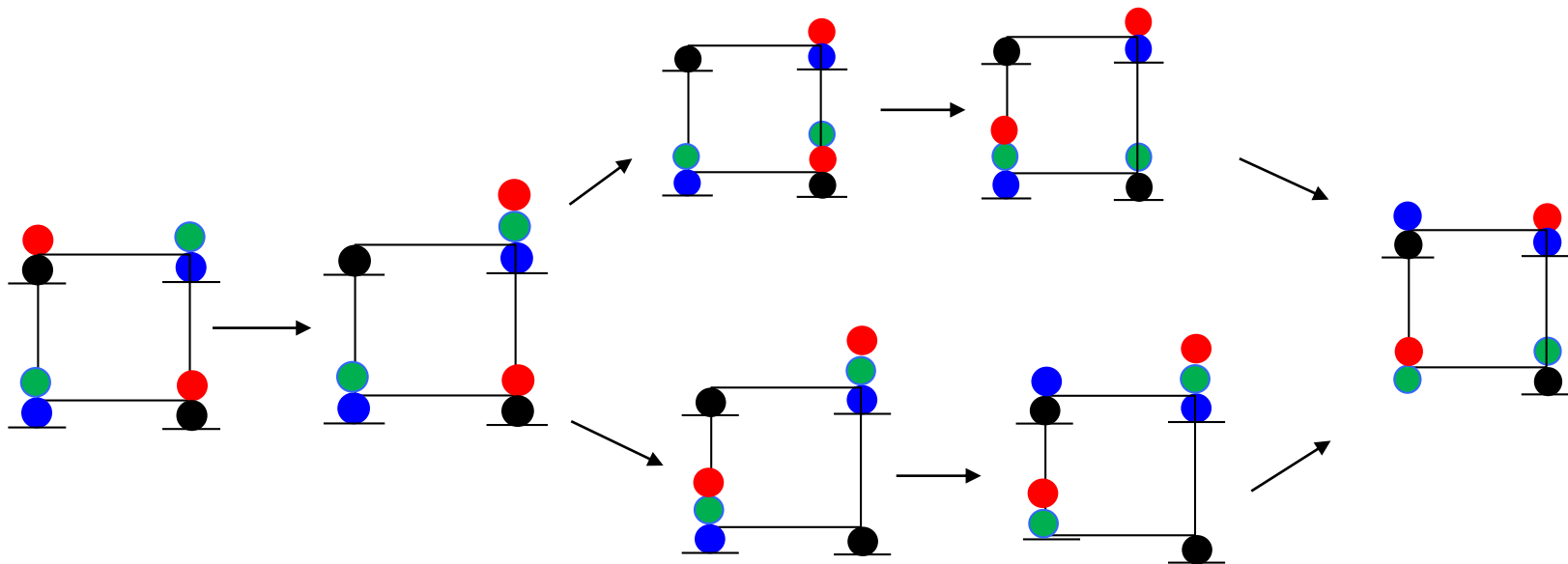
- Anomalous dimension via structural factor scaling

$$D^2(L) = L^{-z-\eta} f\left(|U - U_c| L^{\frac{1}{\nu}}\right) \rightarrow z + \eta = 1.86 \pm 0.04$$



# Ring exchange

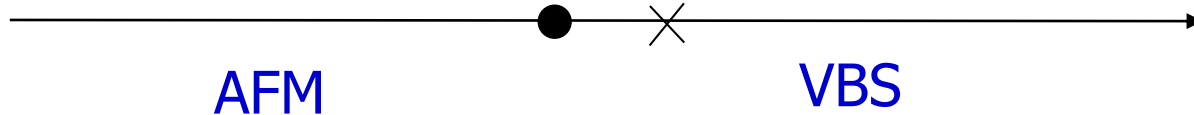
- Difference between 0 and  $\pi$ -flux SU(4) Hubbard model starts from the ring-exchange process: the order of  $t^4/U^3$ .



0 - flux

$t/U = 0$   
Heisenberg

$\pi$  - flux



Slater v.s Mott  
Neel v.s VBS

Convergence of  
itinerancy and local  
Mottness?

**SU(N)**  
**Mott Physics**

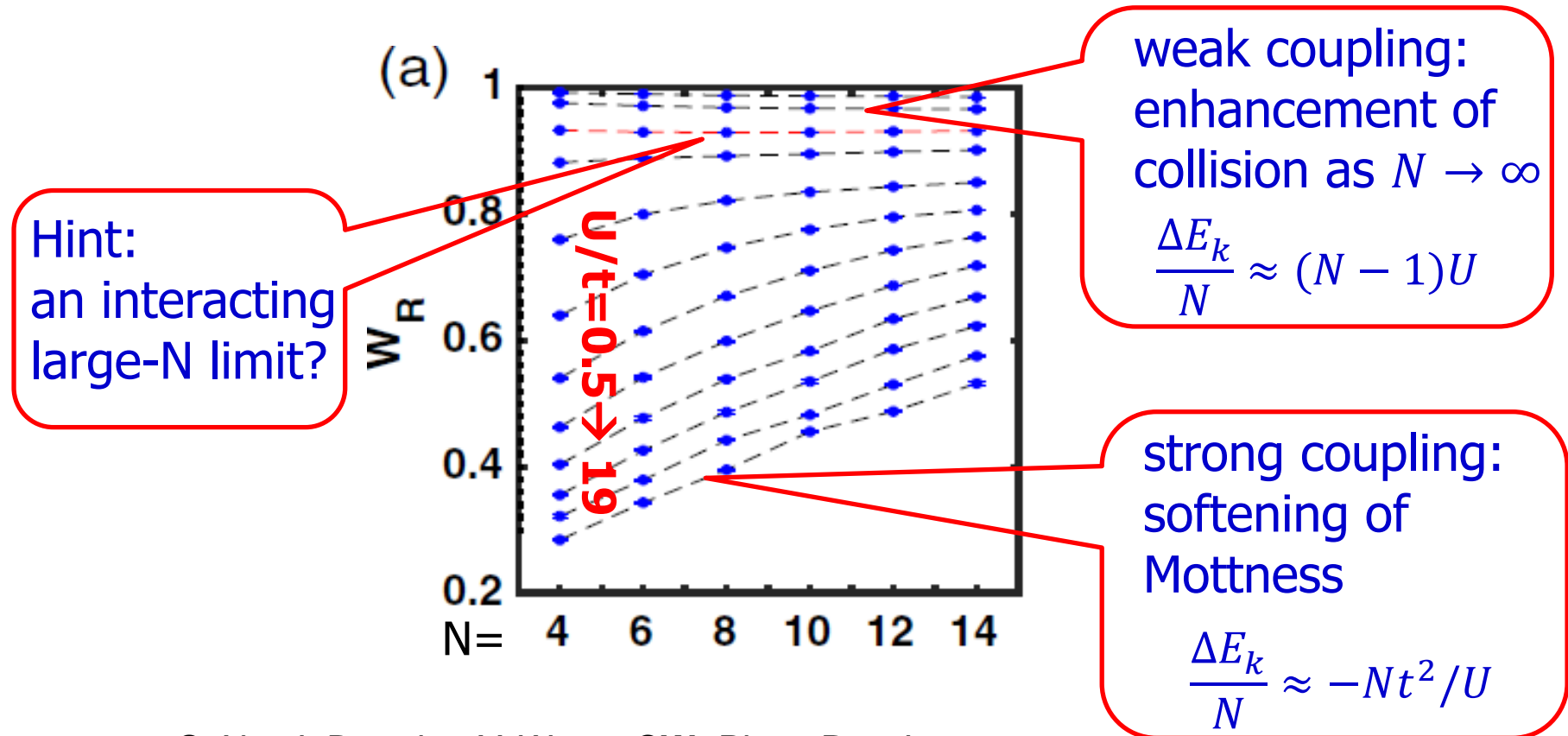
Color Magnetism



# Convergence of itinerancy and locality

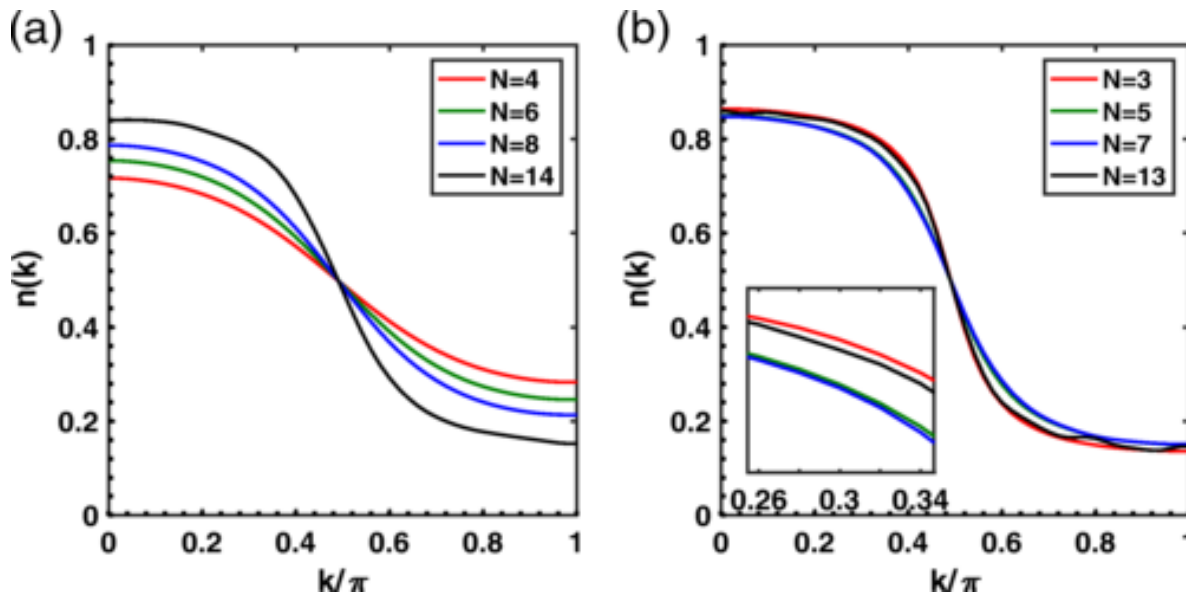
1D half-filled SU(N) Hubbard model: QMC study

relative band width:  $W_R = \langle E_K(U) \rangle / \langle E_k^0 \rangle$



# Fermi distribution in the strong coupling regime

- Sharpening  $n_f(k)$  as  $N$  (even) increases
- $n_f \neq$  the ideal Fermi distribution as  $N \rightarrow \infty$

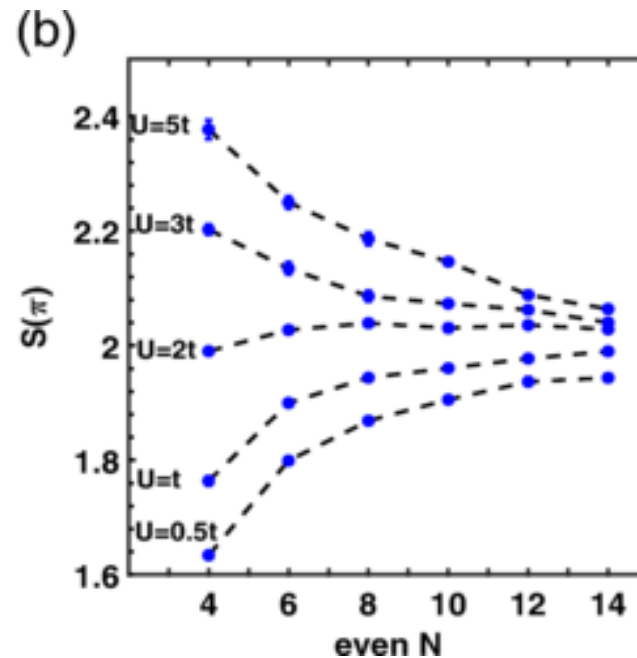
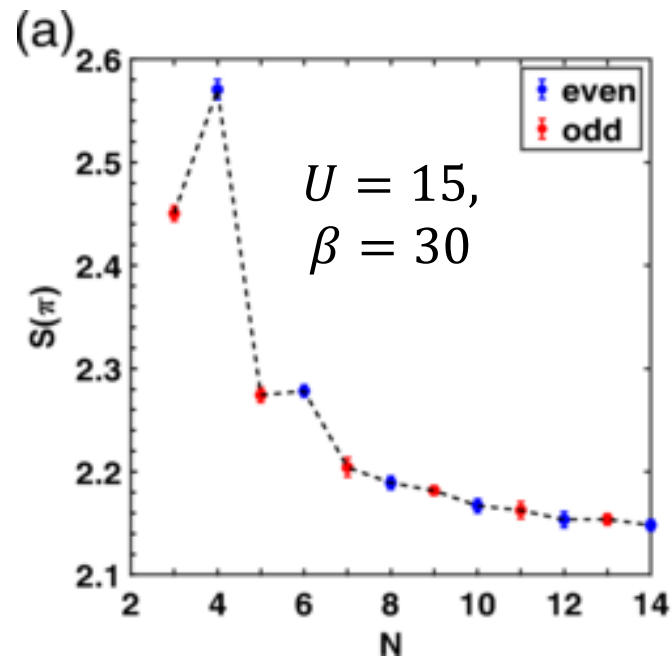


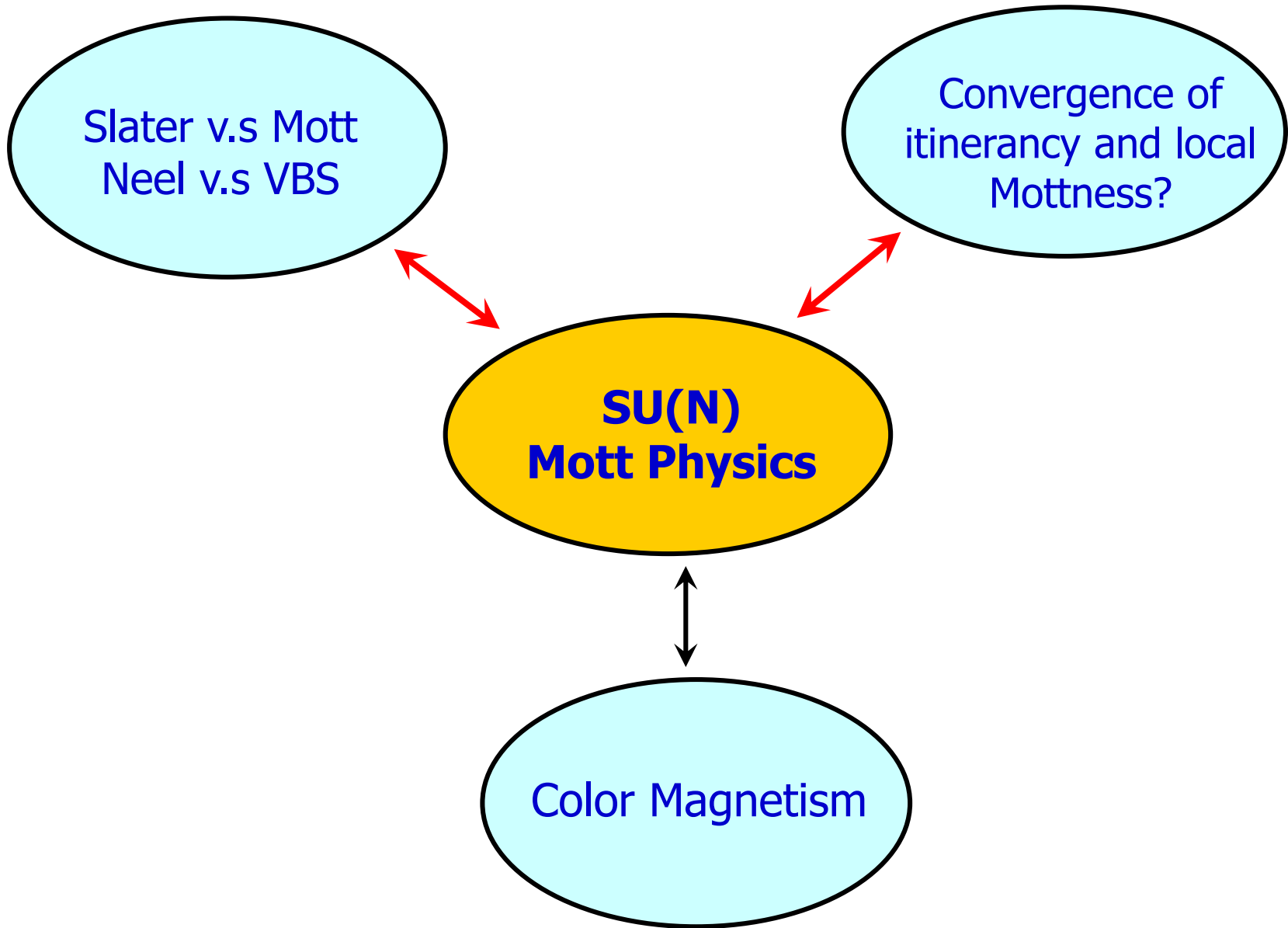
$$U = 15, \quad \beta = 30$$



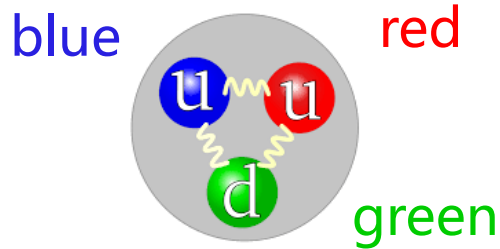
# The AF structure factor

- Opposite dependences of  $S(\pi)$  on  $N$  from the strong (local moment) and weak (itinerancy) interaction regimes.

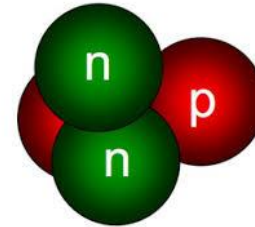




# 1/4-filling of SU(4) -- "color magnetism"



高能：重子（质子） -  
3夸克色单态

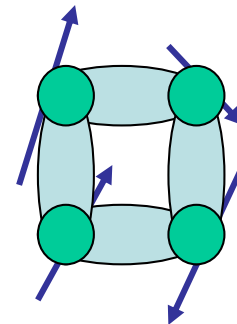


核物理：α-粒子（两  
质子，两中子）。

- 4 sites → an SU(4) singlet (Each site belongs to the fundamental Rep. )

baryon-like 
$$\frac{\varepsilon_{\alpha\beta\gamma\delta}}{4!} \psi_{\alpha}^{+}(1)\psi_{\beta}^{+}(2)\psi_{\gamma}^{+}(3)\psi_{\delta}^{+}(4)|0\rangle$$

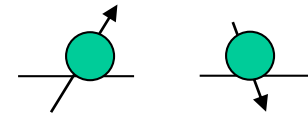
Bossche et. al., Eur. Phys. J. B 17, 367 (2000).



# Sp(4) Heisenberg model at 1/4-filling

- Spin exchange: bond singlet ( $J_0$ ), quintet ( $J_2$ ).

$$H_{ex} = \sum_{\langle ij \rangle} -J_0 Q_0(ij) - J_2 Q_2(ij)$$



$$J_0 = 4t^2 / U_0, J_2 = 4t^2 / U_2, J_1 = J_3 = 0$$

$$\frac{3}{2} \times \frac{3}{2} = 0+2+1+3$$

- SO(5) or Sp(4) explicitly invariant form:

$$H_{ex} = \sum_{ij} \frac{J_0 + J_2}{4} L_{ab}(i)L_{ab}(j) + \frac{-J_0 + 3J_2}{4} n_a(i)n_b(j) \quad a, b = 1 \sim 5$$

$L_{ab}$ : 3 spins + 7 spin-octupole tensors;  $n_a$ : spin-quadrupole operators;  
 $L_{ab}$  and  $n_a$  together form the 15 SU(4) generators.

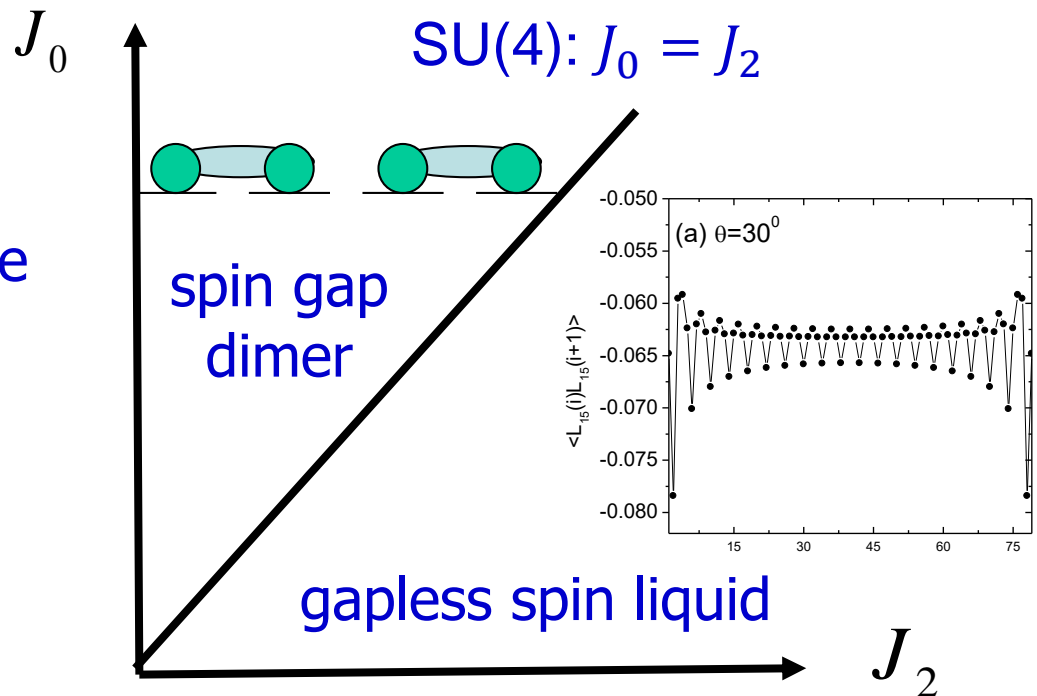
# 1D lattice (one particle per site)

- Phase diagram obtained from bosonization + DMRG.

- Gapped spin dimer phase at  $J_0 > J_2$ .

- Gapless spin liquid phase at  $J_0 \leq J_2$ .

- Spin correlation exhibits 4-site periodicity of oscillations.



# Sp(4) magnetism: a four-site problem

- Bond spin singlet:

- Plaquette SU(4) singlet:

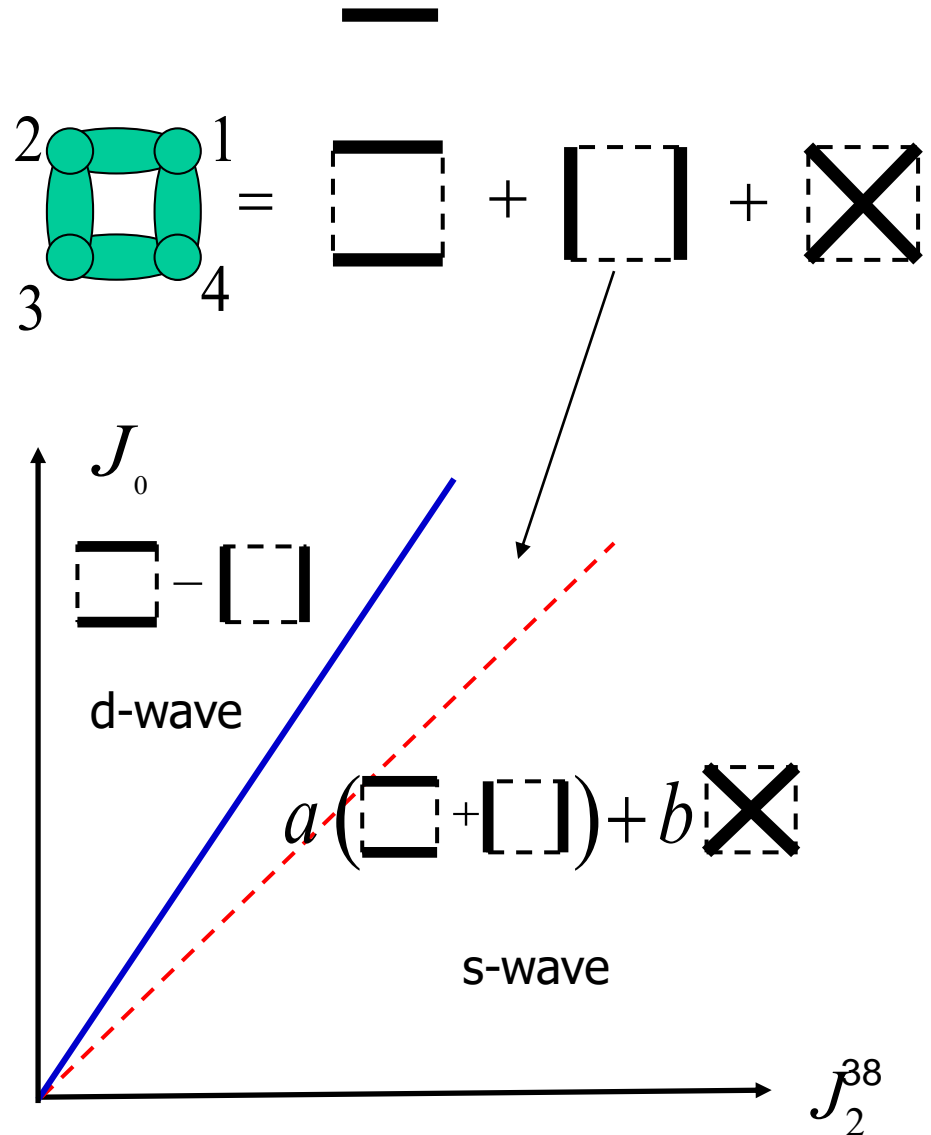
$$\frac{\epsilon_{\alpha\beta\gamma\delta}}{4!} \psi_{\alpha}^+ \psi_{\beta}^+ \psi_{\gamma}^+ \psi_{\delta}^+ |0\rangle$$

4-body EPR state; no bond orders

- Level crossing:

d-wave to s-wave

- Hint to 2D?



# Unsolved difficulty: 2D phase diagram

- $J_2=0$ , Neel ordering obtained by QMC.

K. Harada et. al. PRL 90, 117203, (2003).

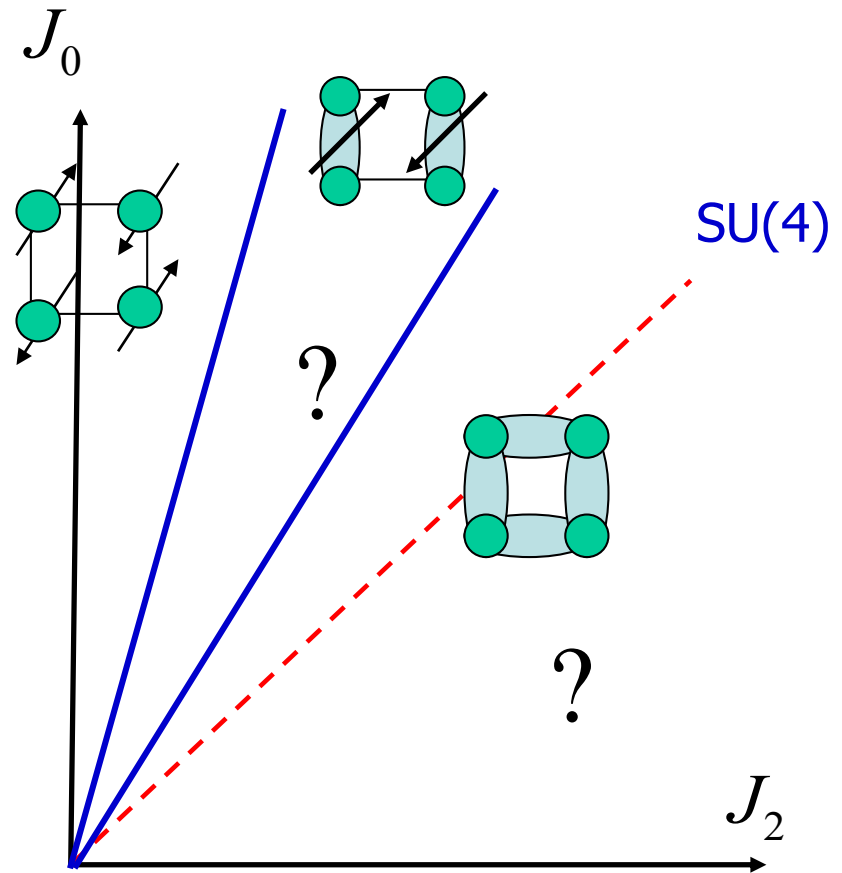
- $J_2>0$ , no conclusive results!  
Difficult both analytically and numerically.

2D Plaquette ordering at the SU(4) point?

Exact diagonalization on a 4\*4 lattice

Bossche et. al., Eur. Phys. J. B 17, 367 (2000).

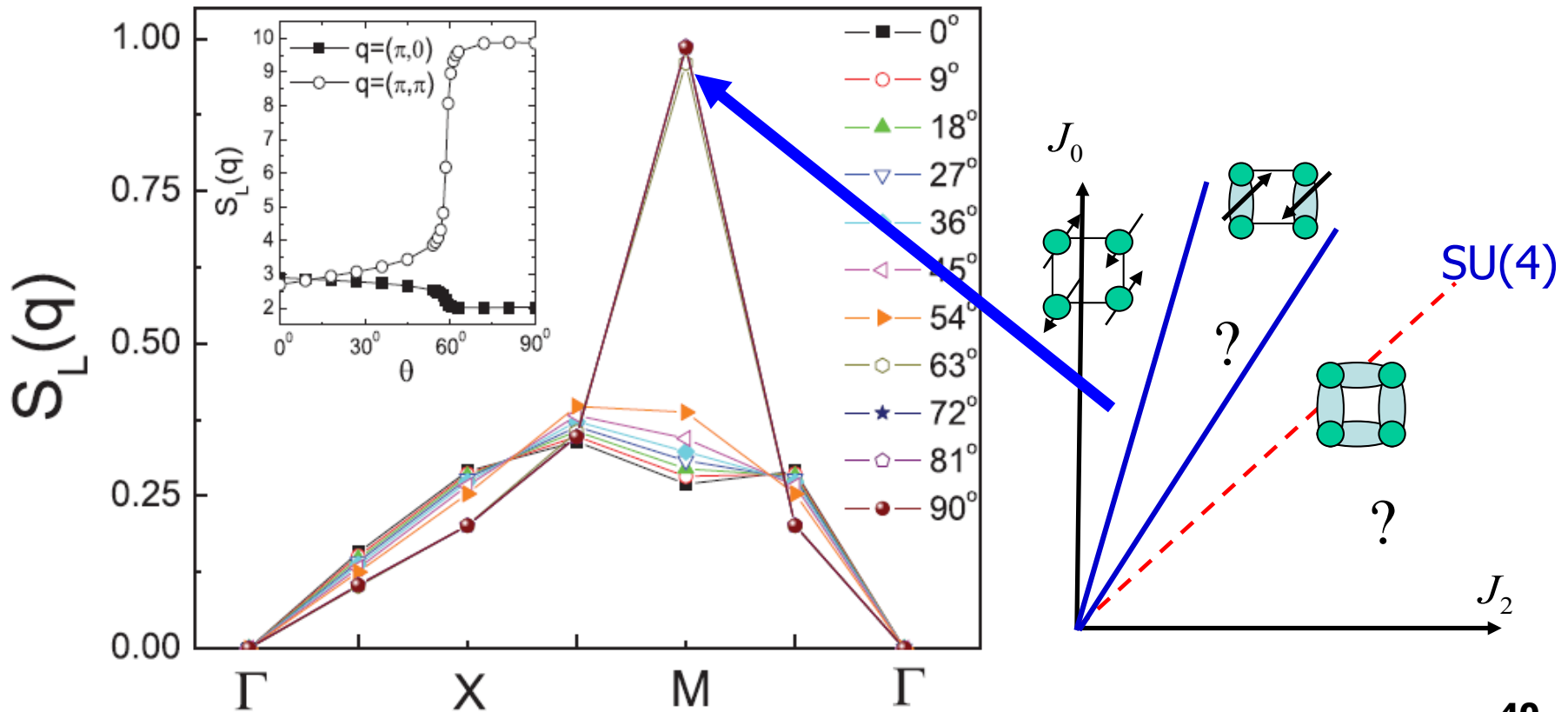
- Phase transitions as  $J_0/J_2$ ?



# 4x4 Exact diag. (I): Neel correlation

- Spin structure form factor peaks at  $(\pi, \pi)$  at  $\theta > 60^\circ$ , indicating strong Neel correlation.

$$S_L(q) \sim \sum_{1 \leq a < b \leq 5} \sum_{i,j} \langle L_{ab}(i) L_{ab}(j) \rangle e^{i\vec{q} \cdot (\vec{r}_i - \vec{r}_j)}$$





# 4x4 Exact Diag. (II): Dimer correlation

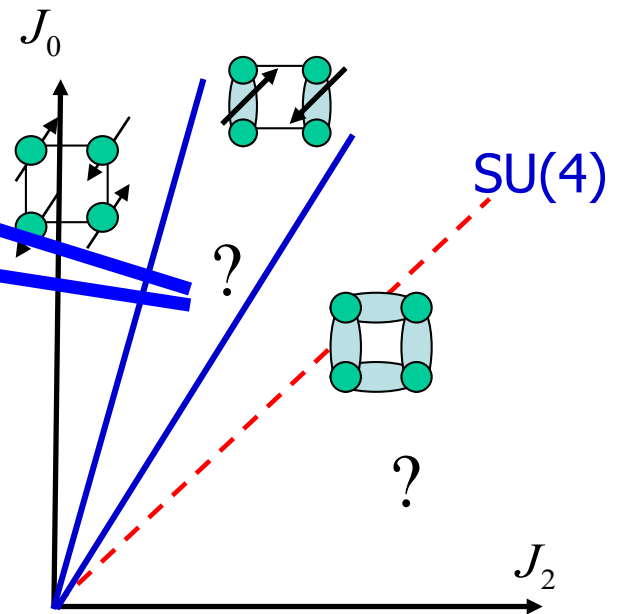
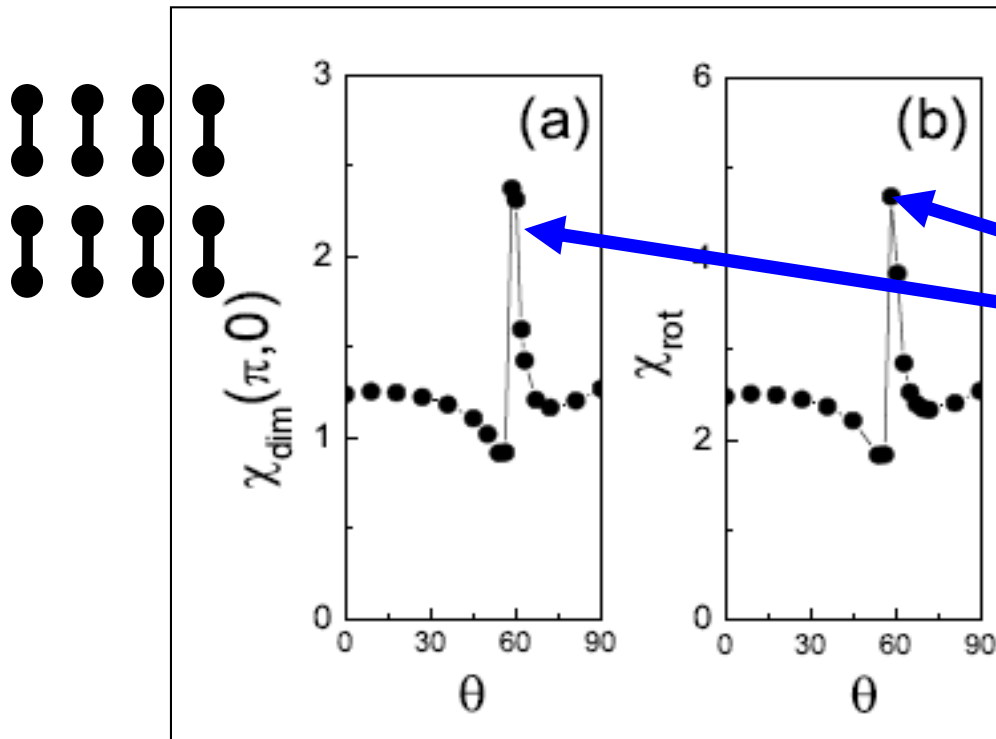
- **Susceptibility:**  $H(\delta) = H_{exc} + \delta^* H_{perp}$      $E(\delta) = E(0) - \frac{1}{2} \chi \delta^2$ ,

- **a) Break translational symm:**

- **b) Break rotational symm:**

$$H_{pert} = \sum_i \cos(\vec{Q} \cdot \vec{r}_i) H_{ex}(i, i+x),$$

$$H_{pert} = \sum_i H_{ex}(i, i+x) - H_{ex}(i, i+y)$$

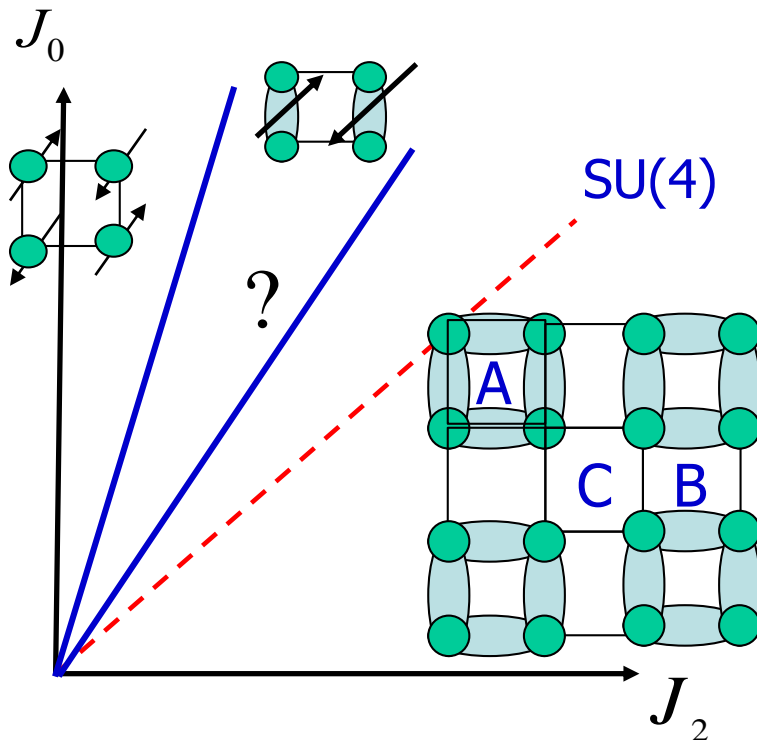


# 4x4 Exact Diag. (III): Plaquette formation?

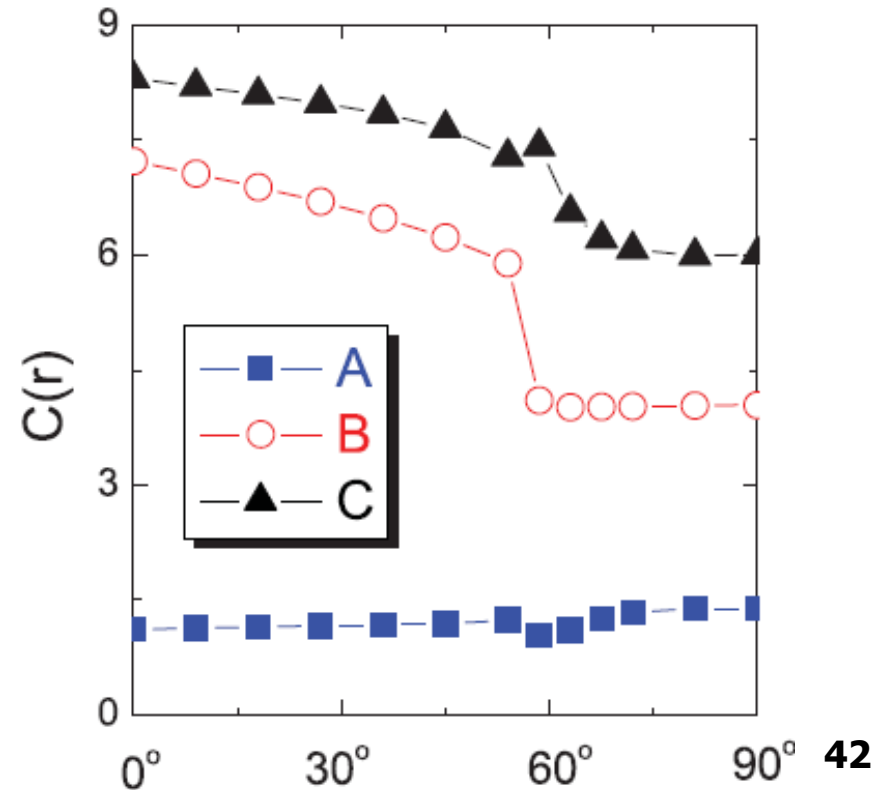
- Local Casimir; analogy to total spin of SU(2).

$$C(r) \sim \left\langle \sum_{1 \leq a < b \leq 5} \left\{ \sum_{i \in \text{plaquette } r} L_{ab}(i) \right\}^2 \right\rangle$$

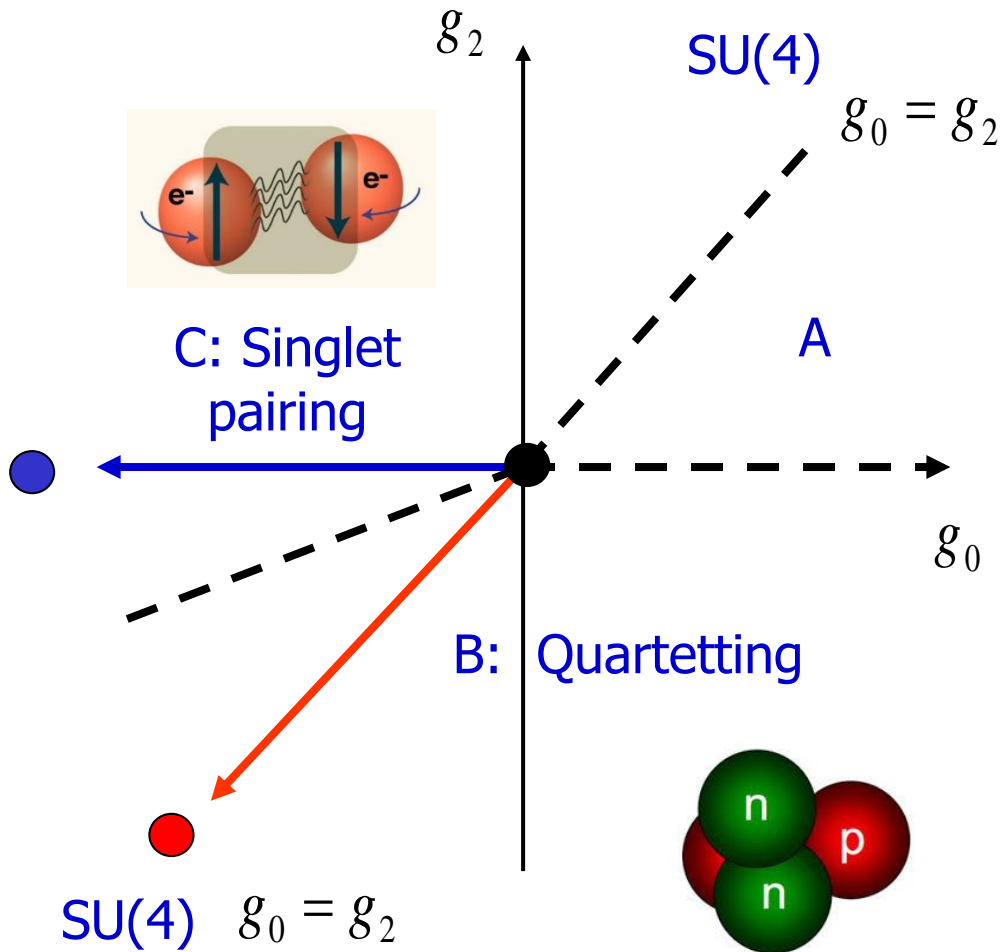
$C(r) \rightarrow 0$ : singlet



Open boundary condition



# Pairing v.s. quartetting (RG 1 loop)



- (A) Luttinger liquid, spin-charge separation.

- Two spin gap phases: (B) quartetting, (C) pairing.

- Ising duality between (B) and (C).

# Digression: itinerant FM based on Hubbard model

An entire phase of ground state itinerant FM

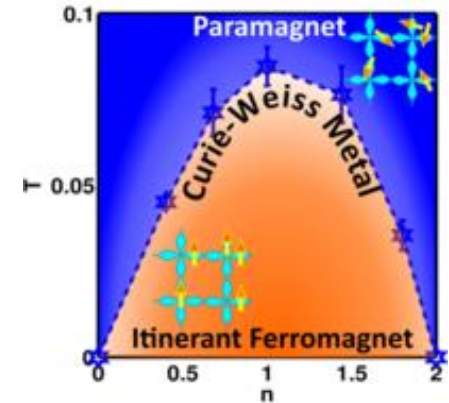
Y. Li, E. H. Lieb, **CW**, PRL 112, 217201 (2014).

S. Xu, Y. Li, and **CW**, PRX 5, 021032, (2015).

QMC study to Curie-Weiss metal (sign problem-free)

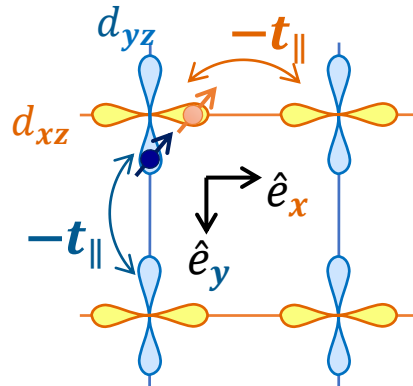
The first proof to our knowledge.

A simple and quasi-realistic model



Curie-Weiss susceptibility v.s. metallic compressibility

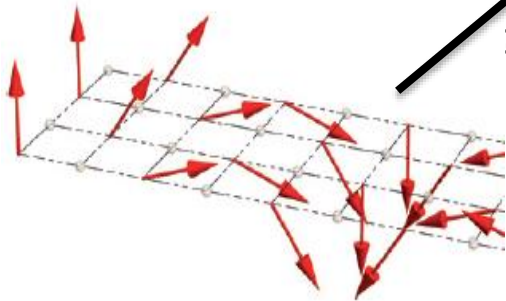
Mechanism and Experiments



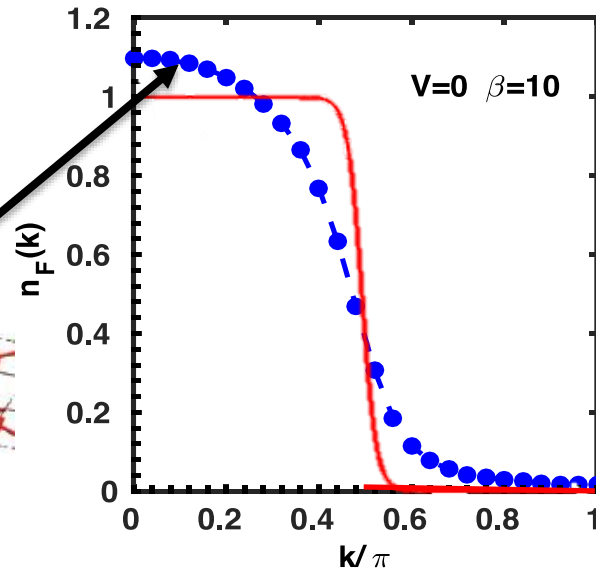
1. Strong correlation
2. Hund's rule

# Momentum space picture for CW metal

Domain fluctuations  
c.f. pseudogap phase  
of high  $T_c$



$$n_F(k) = n_{\uparrow}(k) + n_{\downarrow}(k)$$



$$T_0/t \approx 0.08$$

$$\text{At } k \rightarrow 0, n_{\uparrow}(k) = n_{\downarrow}(k) \approx 0.54 \ll 1$$

- Exclusion in momentum space  $\langle n_{k\uparrow} n_{k\downarrow} \rangle \approx 0$
- Large entropy capacity  $\rightarrow$  Wilson ratio  $\chi T / C = ?$

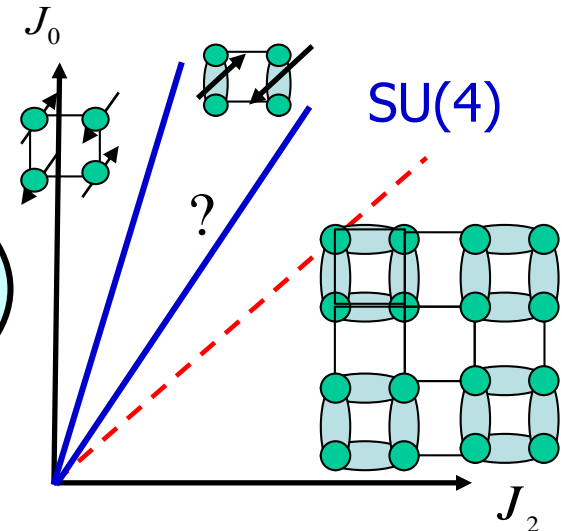
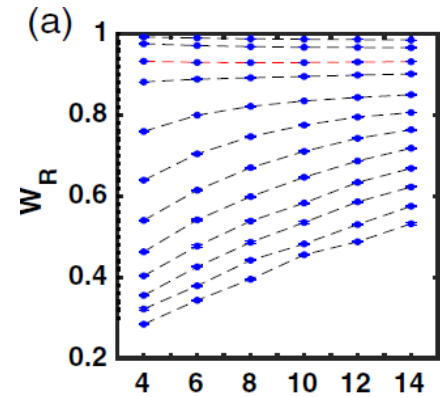
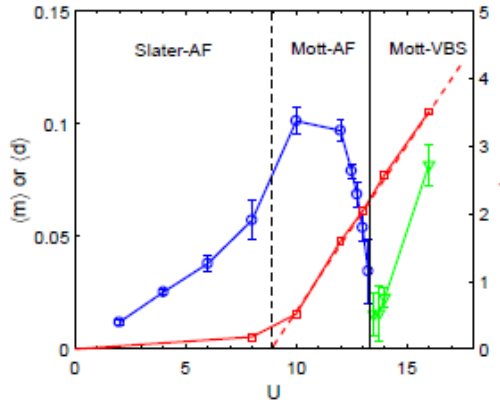
# Summary

Slater v.s Mott  
Neel v.s VBS

Convergence of  
itinerancy and local  
Mottness?

**SU(N)  
Mott Physics**

Color Magnetism



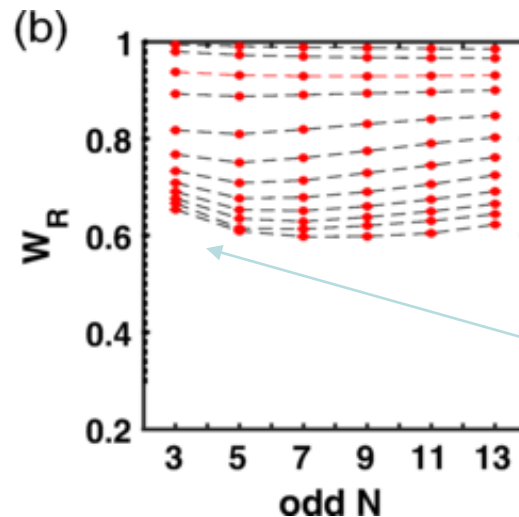
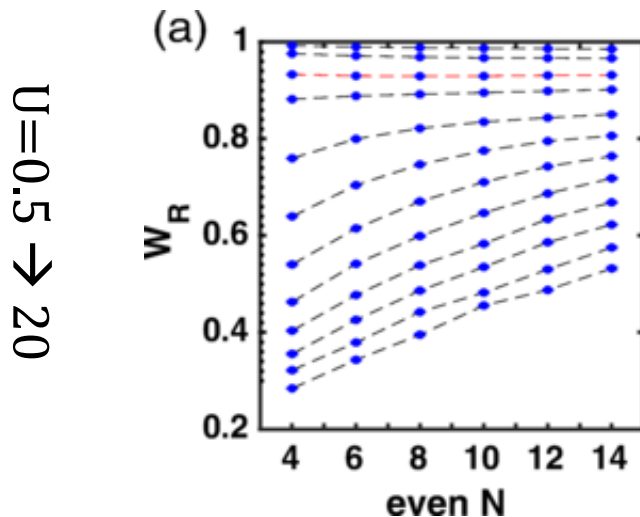
# How do interaction effects scale with N?

- 1D SU(N) Hubbard models: U fixed, particle # per site N/2, i.e., ( $k_f = \pi/2$ , half-filling).

- Large U limit  $\rightarrow$  softening of Mottness  $\rightarrow \frac{\Delta E_k}{N} \approx -Nt^2/U$   
 Small U limit  $\rightarrow$  enhance of collision  $\rightarrow \frac{\Delta E_k}{N} \approx (N-1)U$

- Relative band width:

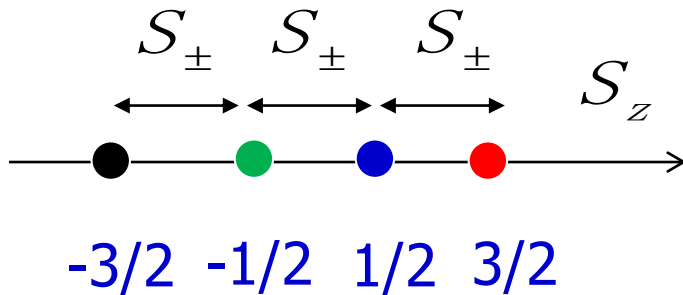
$$W_R = \frac{E_K(U)}{E_K(U=0)}$$



Preexisting charge fluctuation at odd N's

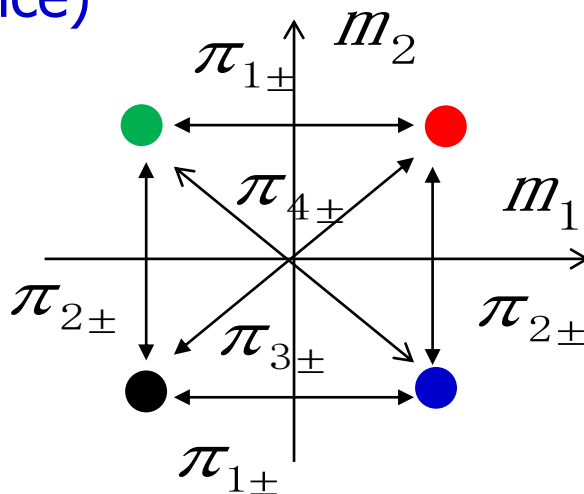
# Two views of spin quartet (weight diagrams of Lie algebra)

Solid: SU(2) (1D lattice)



- A high rank spinor Rep. of a small group.
- Off-diagonal operator: (fluctuation)  $S_{\pm}$

Cold fermions Sp(4) or SO(5) (2D lattice)



- The fundamental spinor Rep of a large group.
- Much more off-diagonal operators.

$$\pi_{1\pm}, \pi_{2\pm}, \pi_{3\pm}, \pi_{4\pm}$$