Quaternionic analyticity and SU(2) Landau Levels in 3D

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Outline

- Introduction: complex number → quaternion.

- **Quaternionic analytic** Landau levels in 3D/4D.

  Analyticity : a useful rule to select wavefunctions for non-trivial topology.

  Cauchy-Riemann-Fueter condition.

  3D harmonic oscillator + SO coupling.

- 3D/4D Landau levels of **Dirac fermions**: complex quaternions.

  An entire flat-band of half-fermion zero modes (anomaly?)

The birth of "*i*": not from $x^2 = -1$

- Cardano formula for the cubic equation.

$$x^3 + px + q = 0 \quad \Rightarrow \quad x_1 = c_1 + c_2,\ x_{2,3} = c_1 e^{\pm i\frac{2}{3}\pi} + c_2 e^{\mp i\frac{2}{3}\pi}$$

$$c_{1,2} = \sqrt[3]{-\frac{q}{2} \pm \sqrt{\Delta}}$$ discriminant: $$\Delta = \left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3$$

- Start with **real** coefficients, and end up with three **real** roots, but no way to avoid **"i"**.

$$x^3 - 3x = 0 \quad \Rightarrow \quad p = -3, q = 0,\ \sqrt{\Delta} = \sqrt{-1}, \quad \Rightarrow \quad x_1 = 0,\ x_{2,3} = \pm\sqrt{3}$$

The beauty of "complex"

- Gauss plane: 2D rotation (angular momentum)
- Euler formula: $e^{i\theta} = \cos\theta + i\sin\theta$ (U(1) phase: optics, QM)
- Complex analyticity: (2D lowest Landau level)

$$\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y} = 0 \Rightarrow \boxed{\frac{1}{2\pi i}\oint \frac{1}{z - z_0}\,dz\; f(z) = f(z_0)}$$

- Algebra fundamental theorem;
  Riemann hypothesis – distributions of prime numbers, etc.
- Quan Mech: "i" appears for the first time in a wave equation.

Schroedinger Eq: $i\hbar\frac{\partial}{\partial t}\psi = H\psi$

Further extension: quaternion (Hamilton number)

- Three imaginary units i, j, k.

$$q = x + yi + zj + uk \qquad i^2 = j^2 = k^2 = -1$$

- Division algebra: $ab = 0 \Leftrightarrow a = 0, \text{ or, } b = 0$

- 3D rotation: non-commutative.

$$ij = -ji = k; \quad jk = -kj = i; \quad ki = -ik = j$$

- Quaternion-analyticity (Cauchy-Futer integral)

$$\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y} + j\frac{\partial f}{\partial z} + k\frac{\partial f}{\partial u} = 0 \quad \Rightarrow \quad \frac{1}{2\pi^2}\oiiint \frac{1}{|q-q_0|^2 (q-q_0)} Dq \; f(q) = f(q_0)$$

Quaternion plaque: Hamilton 10/16/1843

Brougham bridge, Dublin

$$i^2 = j^2 = k^2 = ijk = -1$$

3D rotation as 1st Hopf map

- Rotation axis $\hat{\Omega}$, rotation angle: γ.
- $\hat{\Omega}$: ⟷ imaginary unit: $\omega(\hat{\Omega}) = i \sin\theta\cos\phi + j \sin\theta\sin\phi + k\cos\theta$

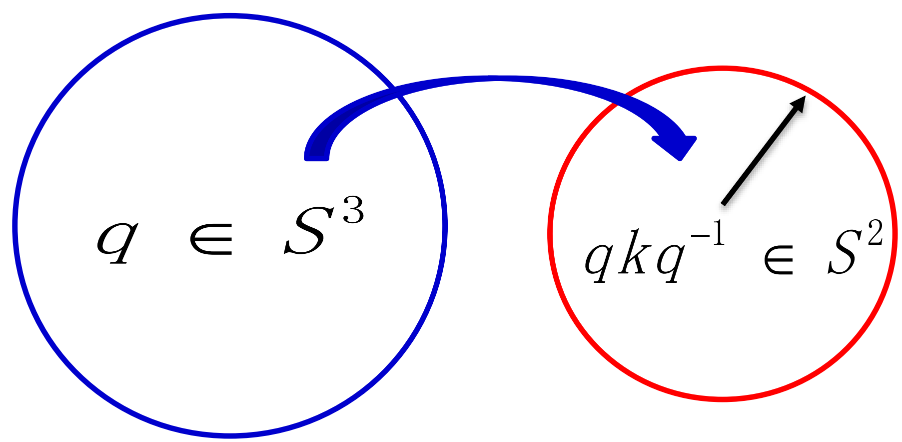
rotation R ⟷ unit quaternion q: $q = \cos\frac{\gamma}{2} + \omega(\hat{\Omega})\sin\frac{\gamma}{2} \in S^3$

- 3D vector r ⟷ imaginary quaternion. $\vec{r} \Rightarrow xi + yj + zk$
- 3D rotation ⟷ Hopf map S3→ S2.

1st Hopf map

$$\vec{r} = \hat{z} = k$$

$$\vec{r}' = R(\vec{r}) = qkq^{-1}$$



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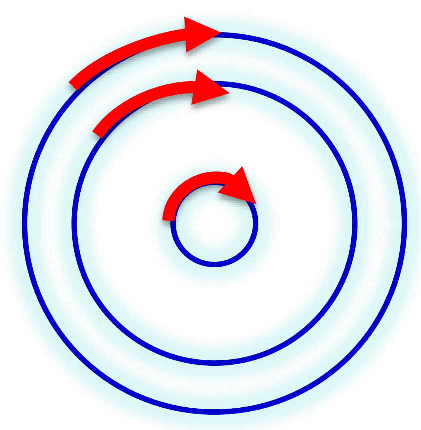
Review: 2D Landau level in the symmetric gauge

symmetric gauge: $$H_{2DLL} = \frac{1}{2M}(\vec{P} - \frac{e}{c}\vec{A})^2, \quad \vec{A} = \frac{B}{2}\hat{z}\times\vec{r}$$



Lowest Landau level wavefunction:
complex analyticity

$$(a_x + i a_y)\psi_{LLL}(z,\bar{z}) = 0, \ (z = x + iy)$$

$$a_x = \frac{1}{\sqrt{2}}(x + ip_x) \qquad a_y = \frac{1}{\sqrt{2}}(y + ip_y)$$

$$\psi_{LLL}(z,\bar{z}) = f(z,\bar{z}) e^{-\frac{z\bar{z}}{4 l_B^2}},$$
$$\partial_{\bar{z}} f(z,\bar{z}) = 0$$

$\Rightarrow$

$$f(z,\bar{z}) = f(z)$$
$$f(z) = 1, z, z^2, \ldots, \quad z^m, \ldots, \qquad (m \geq 0)$$

Advantages of Landau levels (2D)

- Simple, explicit and elegant.

- **Complex analyticity → selection of non-trivial WFs.**

1. The 2D ordinary QM WF $\psi(x, y)$ belongs to real analysis
2. Cauchy-Riemann condition → complex analyticity (chirality).
3. Chirality is physically imposed by the B-field.

- Analytic properties facilitate the construction of Laughlin WF.

$$\psi_{sym}(z_1, z_2, \ldots .z_n) = \prod_{i<j} (z_i - z_j)^3 e^{-\sum_i \frac{|z_i|^2}{4 l_B^2}}$$

Pioneering Work: LLs on 4D-sphere ---Zhang and Hu

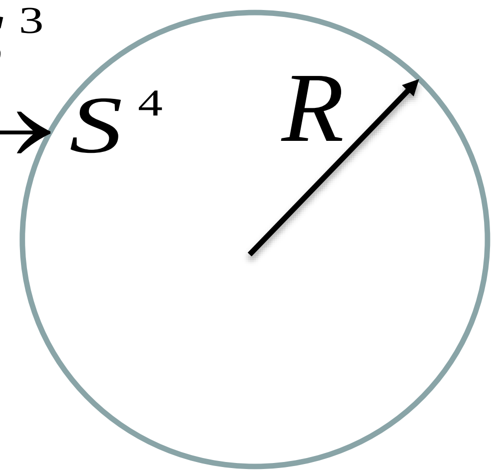
Science 294, 824 (2001).

- Particles couple to the SU(2) gauge field on the S^4 sphere.

$$H = \frac{\hbar^2}{2MR^2} \sum_{1 \le a < b \le 5} \Lambda_{ab}^2, \quad \Lambda_{ab} = x_a(-i\partial_b + A_b) - x_b(-i\partial_a + A_a)$$

- Second Hopf map. The spin value $I \propto R^2$.

$$S^7 \xrightarrow{S^3} S^4$$



$$x_a = \psi_\alpha^+ \Gamma_{\alpha\beta}^a \psi_\beta, \quad n_i = u_\alpha^+ \sigma_{i,\alpha\beta} u_\beta$$

- Single particle LLLs

$$\langle x_a, n_i \mid m_1 m_2 m_3 m_4 \rangle = \psi_1^{m_1} \psi_2^{m_2} \psi_3^{m_3} \psi_4^{m_4}$$

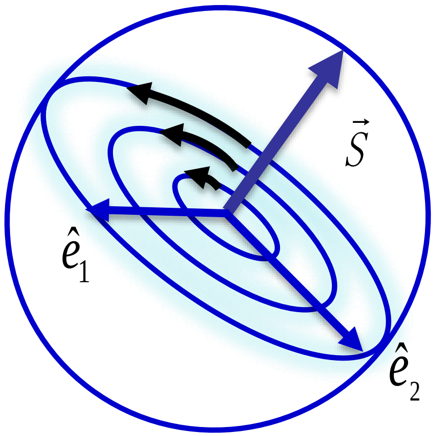
→ **4D integer and fractional TIs with time reversal symmetry**
→ **Dimension reduction to 3D and 2D TIs** (Qi, Hughes, Zhang)**.**

Our recipe

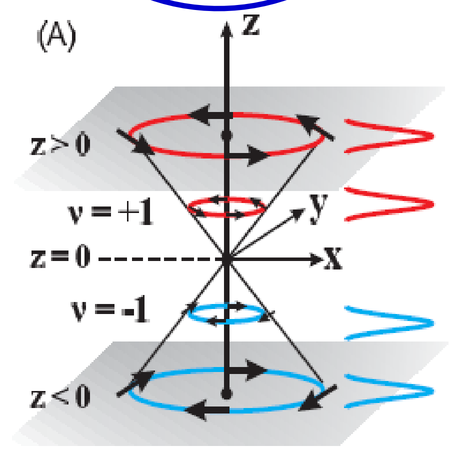
1. 3D harmonic wavefunctions.

2. Selection criterion: quaternionic analyticity (physically imposed by SO coupling).

Symmetric-like gauge (3D quantum top):
Complex analyticity in a flexible e_1-e_2 plane with chirality determined by S along the e_3 direction.



Landau-like gauge: spatial separation of 2D Dirac modes with opposite helicites.



Generalizable to higher dimensions.

2D LLs in the symmetric gauge

$$H_{2DLL} = \frac{1}{2M}(\vec{P} - \frac{e}{c}\vec{A})^2, \quad \vec{A} = \frac{B}{2}\hat{z} \times \vec{r}$$

- 2D LL Hamiltonian = 2D harmonic oscillator (HO)+ orbital Zeeman coupling.

$$H_{2D\,LL} = \frac{p^2}{2M} + \frac{1}{2}M\omega^2 r^2 \mp \omega L_z, \quad \omega = \frac{e/B/}{2Mc}, \quad l_B = \sqrt{\frac{hc}{eB}}$$

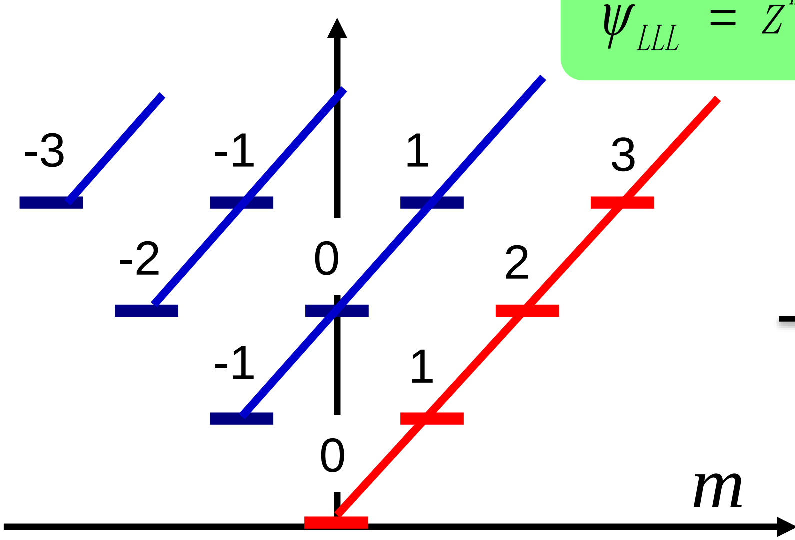
- H_{2DLL} has the same set of eigenstates as 2D HO.

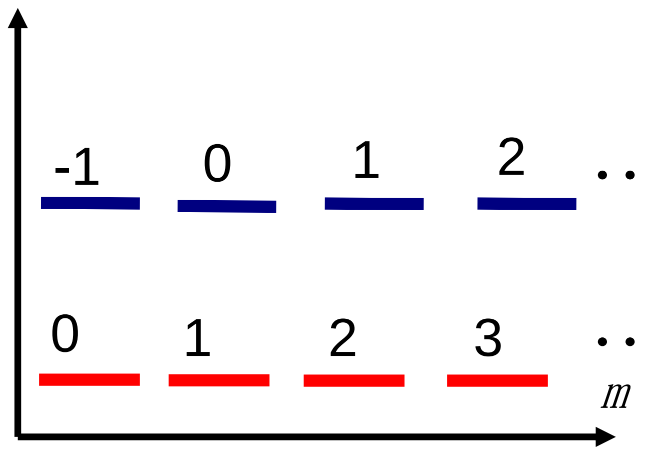
Organization → non-trivial topology

$$E_{2D,\,HO}/(\hbar\omega) = 2n_r + |m| + 1$$

$$\psi_{LLL} = z^m e^{-|z|^2/(2l_B^2)}$$

$$E_{2D,LL}/(\hbar\omega) = 2n_r$$





- If viewed horizontally, they are topologically trivial.

- If viewed along the diagonal line, they become LLs.

$$E_{Zeeman}/(\hbar\omega) = -m$$

3D – Aharanov-Casher potential !!

- The SU(2) gauge potential:

$$2D: \quad \vec{A} = \frac{1}{2} B\hat{z} \times \vec{r} \quad \Longrightarrow \quad 3D: \quad \vec{A}_{\alpha\beta} = \frac{1}{2} g \vec{\sigma}_{\alpha\beta} \times \vec{r}$$

- 3D LL Hamiltonian = 3D HO + spin-orbit coupling.

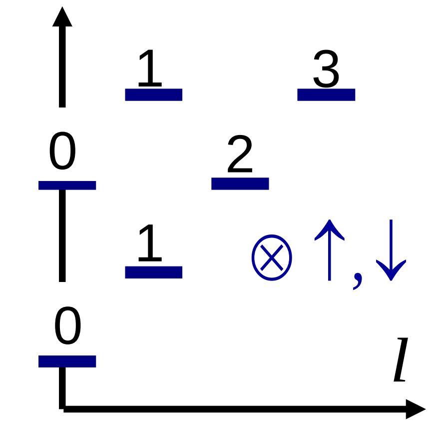
$$\boxed{H_{LL}^{3D} = \frac{P^2}{2M} + \frac{1}{2} M\omega^2 r^2 - \omega \vec{\sigma}_{\alpha\beta} \cdot \vec{L}} \qquad \omega = \frac{|eg|}{2Mc}, \quad l_g = \sqrt{\frac{\hbar c}{|eg|}}.$$

$$= \frac{1}{2M} (\vec{P} - \frac{e}{c} \vec{A}_{\alpha\beta})^2 - \frac{M}{2} \omega^2 r^2$$

- The full 3D rotational symm. + **time-reversal symm**.

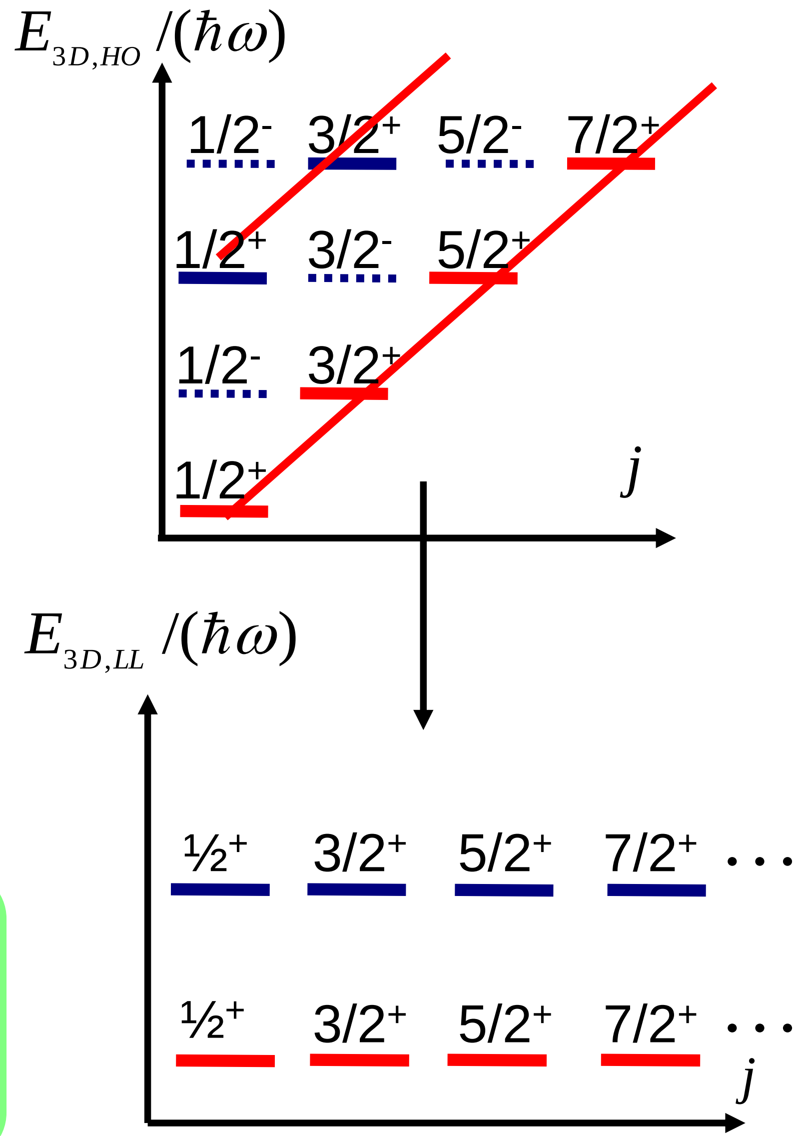
Constructing 3D Landau Levels

$$E_{3D,HO}/(\hbar\omega) = 2n_r + l + \frac{3}{2}$$



SOC : 2 helicity branches

$$j_{\pm} = l \pm \frac{1}{2}.$$



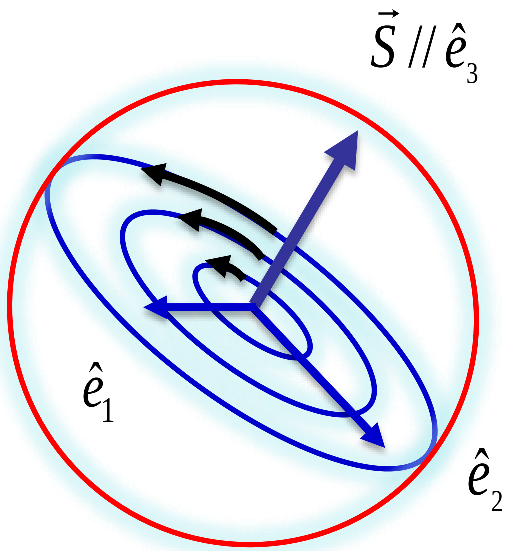
$$H_{LL}^{3D} = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 r^2 - \omega \vec{\sigma} \cdot \vec{L}$$

$$\vec{\sigma} \cdot \vec{L} = \begin{cases} l\hbar & \text{for } j_+ \\ -(l+1)\hbar & \text{for } j_- \end{cases}$$

The coherent state picture for 3D LLL WFs

- The highest weight state $j_z = j_+$. Both L_z and S_z are conserved.

$$\psi^{LLL}_{j_+, j_z = j_+}(\vec{r}) = \begin{pmatrix} (x + iy)^l \\ 0 \end{pmatrix} e^{-r^2/4l_g^2}$$



- Coherent states: spin perpendicular to the orbital plane.

$$\psi^{LLL}_{j_+,\, high}(\vec{r}) = [(\hat{e}_1 + i\hat{e}_2) \cdot \vec{r}]^l \otimes \chi_{\hat{e}_3}$$

- LLLs in N-dimensions: picking up any two axes and define a complex plane with a spin-orbit coupled helical structure.

Comparison of symm. gauge LLs in 2D and 3D

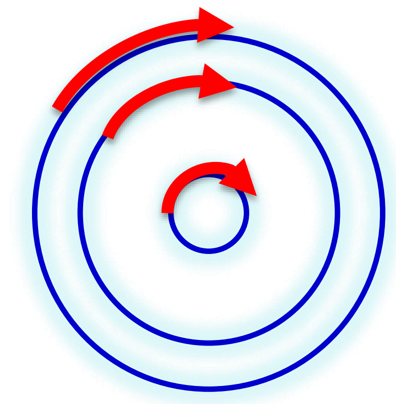
- 1D harmonic levels: real polynomials.

- 2D LLs: complex analytic polynomials.

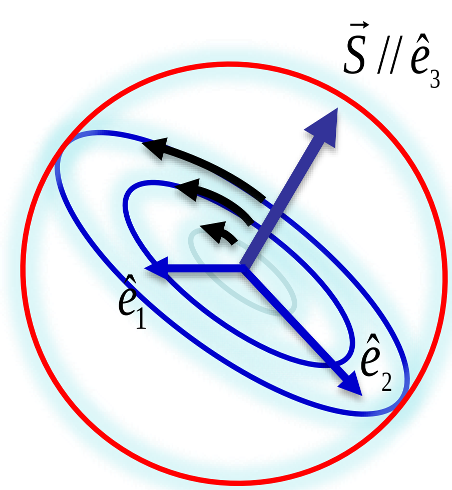
$$\psi_{LLL}^{sym} = z^m e^{-|z|^2/(2l_B^2)}, \qquad z = x + iy, \qquad m \geq 0.$$

- 3D LLs: SU(2) group space → **quaternionic** analytic polynomials.

$$\psi_{j_+,\ high}^{LLL}(r,\Omega) = [(\hat{e}_1 + i\hat{e}_2)\cdot\vec{r}]^l \otimes \chi_{\hat{e}_3} e^{-\frac{r^2}{2l_g^2}}$$



Phase



Right-handed triad

Quaternionic analyticity

- Cauchy-Riemann condition and loop integral.

$$\frac{\partial g}{\partial x} + i\frac{\partial g}{\partial y} = 0 \quad \Rightarrow \quad \frac{1}{2\pi i}\oint \frac{1}{z - z_0} dz\, g(z) = g(z_0)$$

- Fueter condition (left analyticity): f (x,y,z,u) quaternion-valued function of 4-real variables.
- Cauchy-Fueter integrals over closed 3-surface in 4D.

$$\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y} + j\frac{\partial f}{\partial z} + k\frac{\partial f}{\partial u} = 0 \quad \Rightarrow \quad \frac{1}{2\pi^2}\oiiint K(q - q_0)\; Dq\;\; f(q) = f(q_0)$$

$$K(q) = \frac{1}{q\,|\,q\,|^2} = \frac{x - yi - zj - uk}{(x^2 + y^2 + z^2 + u^2)^2}$$

$$D(q) = dy \wedge dz \wedge du - i dx \wedge dz \wedge du + j dx \wedge dy \wedge du - k dx \wedge dy \wedge dz$$

Mapping 2-component spinor to a single quaternion

$$\psi^{LLL}_{j_+,j_z}(r,\hat{\Omega}) = \begin{pmatrix} \psi_{1,j_+,j_z} \\ \psi_{2,j_+,j_z} \end{pmatrix} e^{-\frac{r^2}{4l_G^2}} \quad \rightarrow \quad f_{j,j_z}(x,y,z) = \psi_{1,j_+,j_z} + j\psi_{2,j_+,j_z}$$

- TR reversal: $i\sigma_y\psi^* \rightarrow -fj$; U(1) phase $e^{i\theta}\psi \rightarrow fe^{i\theta}$

SU(2) rotation: $e^{i\frac{\varphi}{2}\sigma_x}\psi \rightarrow e^{k\frac{\varphi}{2}}f;\ e^{i\frac{\varphi}{2}\sigma_y}\psi \rightarrow e^{j\frac{\varphi}{2}}f;\ e^{i\frac{\varphi}{2}\sigma_z}\psi \rightarrow e^{-i\frac{\varphi}{2}}f$

- Reduced Fueter condition in 3D: $\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y} + j\frac{\partial}{\partial z}\right)f(x,y,z) = 0$

- Fueter condition is invariant under rotation $R(\alpha,\beta,\gamma)$.

If *f* satisfies Fueter condition, so does *Rf*.

$$(Rf)(x,y,z) = e^{-i\frac{\alpha}{2}}e^{j\frac{\beta}{2}}e^{-i\frac{\gamma}{2}}f(x',y',z'), \qquad \vec{r}' = R^{-1}\vec{r}$$

Quaternionic analyticity of 3D LLL

• The highest state jz=j is obviously $f_{j_+ j_z = j_+} = (x + iy)^l$ analytic.

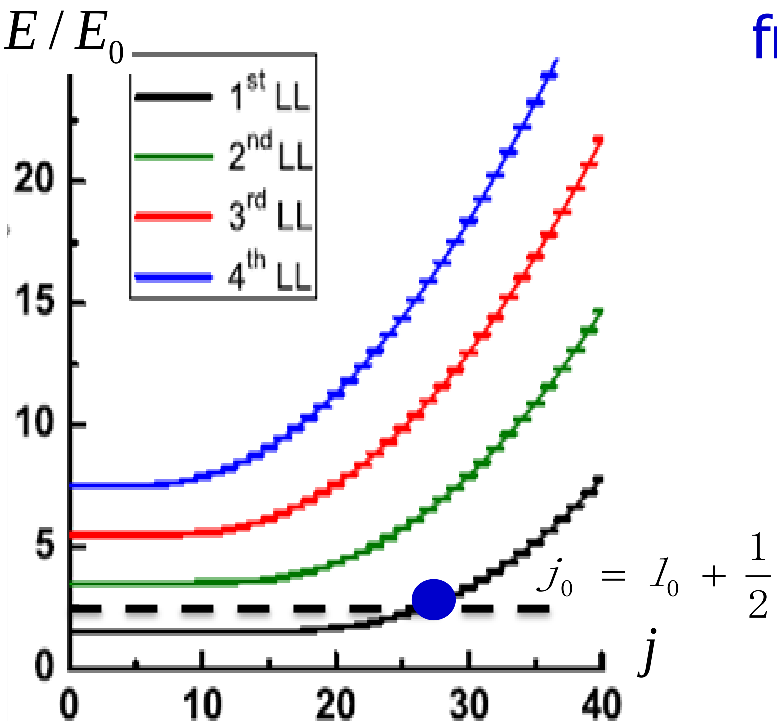
• All the coherent states can be obtained from the highest states through rotations, and thus are also analytic.

• All the LLL states are quaternionic analytic. QED.

• Completeness: Any quaternionic analytic polynomial corresponds to a LLL wavefunction.

$$f = \sum_{j=1/2}^{\infty} \sum_{j_z=-j}^{j} f_{j_+ j_z = m+\frac{1}{2}} c_{jm} = \sum_{j_+=1/2}^{\infty} \sum_{m=0}^{l} f_{j_+ j_z} (c_{jm} - j c_{j-m}) = \sum_{l=0}^{\infty} \sum_{m=0}^{l} f_{j_+ j_z} q_{jm}$$

Helical surface states of 3D LLs



from bulk to surface

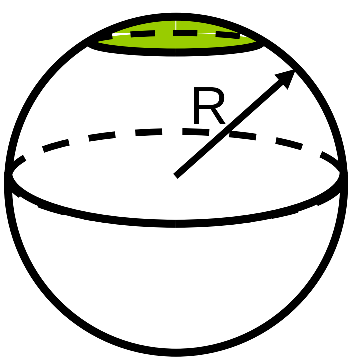
$$H^{3D}_{bulk} = \frac{p^2}{2M} + \cancel{\frac{1}{2}M\omega^2 r^2} - \omega\vec{\sigma}\cdot\vec{L}$$

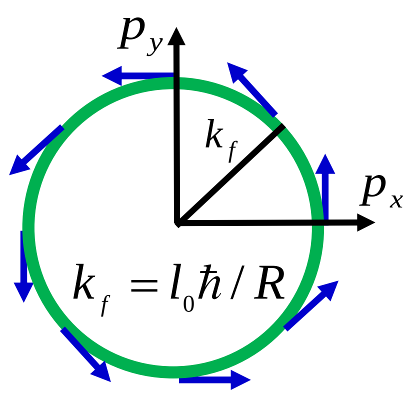
$$\downarrow$$

$$H^{2D}_{surface} = \frac{\hbar^2 l(l+1)}{2MR^2} - \hbar\omega l$$

$$\downarrow \quad l\hbar \rightarrow \vec{\sigma}\cdot\vec{L} = R\hat{e}_r\cdot(\vec{p}\times\vec{\sigma})$$

$$H^{2D}_{plane} = v_f(l-l_0)\hbar/R = v\hat{e}_r\cdot(\vec{p}\times\vec{\sigma}) - \mu$$





- Each LL contributes to one helical Fermi surface.
- Odd fillings yield odd numbers of Dirac Fermi surfaces.

Analyticity condition as Weyl equation (Euclidean)

2D complex analyticity → 1D chiral edge mode

$$\psi_{LLL}(z,\bar{z}) = f(z)e^{-\frac{z\bar{z}}{4l_B^2}}$$

$$\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y} = 0$$

$$\psi(t,x) = f(x-t)$$

$$\frac{\partial \psi}{\partial t} - \frac{\partial \psi}{\partial x} = 0$$

3D: quaternionic analyticity → 2D helical Dirac surface mode

$$\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y} + j\frac{\partial f}{\partial z} = 0$$

$$\frac{\partial \psi}{\partial t} + \sigma_x\frac{\partial \psi}{\partial x} + \sigma_y\frac{\partial \psi}{\partial y} = 0$$

4D: quaternionic analyticity → 3D Weyl boundary mode

$$\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y} + j\frac{\partial f}{\partial z} + k\frac{\partial f}{\partial u} = 0$$

$$\frac{\partial \psi}{\partial t} + \sigma_x\frac{\partial \psi}{\partial x} + \sigma_y\frac{\partial \psi}{\partial y} + \sigma_z\frac{\partial \psi}{\partial z} = 0$$

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- **3D/4D Landau levels of Dirac fermions: complex quaternions.**

  An entire flat-band of half-fermion zero modes (anomaly?)

Review: 2D LL Hamiltonian of Dirac Fermions

$$H_{LL}^{2D} = v_F\{(p_x - \frac{e}{c}A_x)\sigma_x + (p_y - \frac{e}{c}A_y)\sigma_y\}, \qquad \vec{A} = \frac{B}{2}\hat{z}\times\vec{r}$$

- Rewrite in terms of complex combinations of phonon operators.

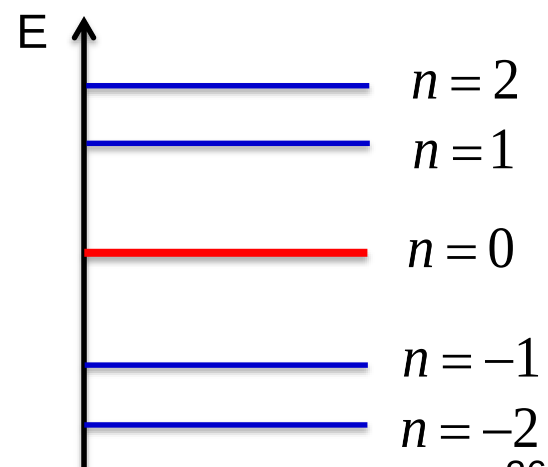
$$\boxed{H_{LL}^{2D} = \frac{\sqrt{2}v_F\hbar}{l_B}\begin{pmatrix} 0 & i(a_x^+ - ia_y^+) \\ -i(a_x + ia_y) & 0 \end{pmatrix}},$$

$$a_i = \frac{1}{\sqrt{2}}(\frac{x_i}{l_B} + i\frac{l_B}{\hbar}p_i), \quad i = x, y.$$

- LL dispersions: $E_{\pm n} = \pm\hbar\omega\sqrt{n}$

- Zero energy LL is a branch of half-fermion modes due to the chiral symmetry.

$$\Psi_{0;m}^{LL} = \begin{pmatrix} z^m \\ 0 \end{pmatrix} e^{-\frac{|z|^2}{4l_B^2}}.$$



3D/4D LL Hamiltonian of Dirac Fermions

$\{1,\ i\}$ $\longleftrightarrow$ 2D harmonic oscillator $\{a_x,\ a_y\}$

$\{1,\ i,\ j,\ k\}$ $(1, -i\sigma_x, -i\sigma_y, -i\sigma_z)$ $\longleftrightarrow$ 4D harmonic oscillator $\{a_u,\ a_x,\ a_y,\ a_z\}$

- "complex quaternion": $a_u - ia_x - ja_y - ka_z$

- 4D Dirac LL Hamiltonian:

$$H_{LL}^{4D\ Dirac} = \frac{\hbar\omega}{2}\begin{pmatrix} 0 & a_u^{+} + ia_x^{+} + ja_y^{+} + ka_z^{+} \\ a_u - ia_x - ja_y - ka_z & 0 \end{pmatrix}$$

$$\Rightarrow \frac{l_0\omega}{2}\begin{pmatrix} 0 & a_u^{+} - i\vec{\sigma}\cdot\vec{a}^{+} \\ a_u + i\vec{\sigma}\cdot\vec{a} & 0 \end{pmatrix}$$

3D LL Hamiltonian of Dirac Fermions

$$H_{LL}^{3D\,Dirac} = \frac{\hbar\omega}{2}\begin{pmatrix} 0 & i\vec{\sigma}\cdot\vec{a}^{+} \\ -i\vec{\sigma}\cdot\vec{a} & 0 \end{pmatrix} = \frac{l_0\omega}{2}\begin{pmatrix} 0 & \vec{\sigma}\cdot(\vec{p}+i\hbar\vec{r}/l_0^2) \\ \vec{\sigma}\cdot(\vec{p}-i\hbar\vec{r}/l_0^2) & 0 \end{pmatrix}$$

- This Lagrangian of non-minimal Pauli coupling.

$$L = \bar{\psi}\{i\hbar(\gamma_0\partial_0 - v\gamma_i\partial_i)\}\psi + \frac{v\hbar}{l_g}\bar{\psi}\sigma_{0i}F^{0i}\psi,$$

$$\sigma_{0i} = -\frac{i}{2}[\gamma_0, \gamma_i], \quad F^{0i} = \frac{x^i}{l_g}.$$

- A related Hamiltonian was studied before under the name of Dirac oscillator, but its connection to LL and topological properties was not noticed.

Benitez, et al, PRL, 64, 1643 (1990)

LL Hamiltonian of Dirac Fermions in Arbitrary Dimensions

- For odd dimensions (D=2k+1).

$$H_{LL}^{D-\mathrm{dim}} = \frac{\hbar\omega}{2}\begin{pmatrix} 0 & i\Gamma_i^{(k)}\cdot a_i^{+} \\ -i\Gamma_i^{(k)}\cdot a_i & 0 \end{pmatrix}$$

- For even dimensions (D=2k).

$$H_{LL}^{D-\mathrm{dim}} = \frac{\hbar\omega}{2}\begin{pmatrix} 0 & \pm a_{2k}^{+} + i\sum_{i=1}^{k}\Gamma_i^{(k-1)}a_i^{+} \\ \pm a_{2k} - i\sum_{i=1}^{k}\Gamma_i^{(k-1)}a_i & 0 \end{pmatrix}$$

A square root problem: $\sqrt{H_{LL}^{3D,Schroedinger}} = H_{LL}^{3D\,Dirac}$

- The square of $H_{LL}^{3D\,Dirac}$ gives two copies of $(H_{LL}^{3D})_{\pm}$ with opposite helicity eigenstates.

$$\frac{(H_{LL}^{3D\,Dirac})^2}{\hbar\omega/2} = \frac{\vec{p}^{\,2}}{2M} + \frac{M}{2}\omega^2\vec{r}^{\,2} + \omega\begin{pmatrix} \vec{L}\cdot\vec{\sigma} + \frac{3}{2}\hbar & 0 \\ 0 & -(\vec{L}\cdot\vec{\sigma} + \frac{3}{2}\hbar) \end{pmatrix}$$

- LL solutions: dispersionless with respect to j. Eigen-states constructed based on non-relativistic LLs.

$$E_{\pm n_r}^{LL} = \pm\hbar\omega\sqrt{n_r},$$

$$\Psi_{\pm n_r;j,l,j_z}^{LL} = \frac{1}{\sqrt{2}}\begin{pmatrix} \psi_{n_r,j_+,l,j_z} \\ \pm i\psi_{n_r-1,j_-,l+1,j_z} \end{pmatrix}.$$

The zeroth LL:

$$\Psi_{0;j,l,j_z}^{LL} = \begin{pmatrix} \psi_{j_+,l,j_z}^{LLL} \\ 0 \end{pmatrix}.$$

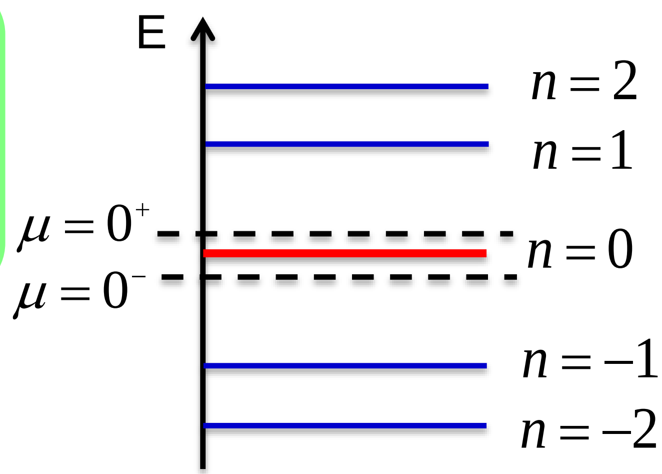
Zeroth LLs as half-fermion modes

• The LL spectra are symmetric with respect to zero energy, thus each state of the zeroth LL contributes $\pm$ ½- fermion charge depending on the zeroth LL is filled or empty.

• For the 2D case, the vacuum charge density is $j_0 = \pm \frac{1}{2}\frac{e^2}{h}B$, known as **parity anomaly**.

G. Semenoff, Phys. Rev. Lett., 53, 2449 (1984).

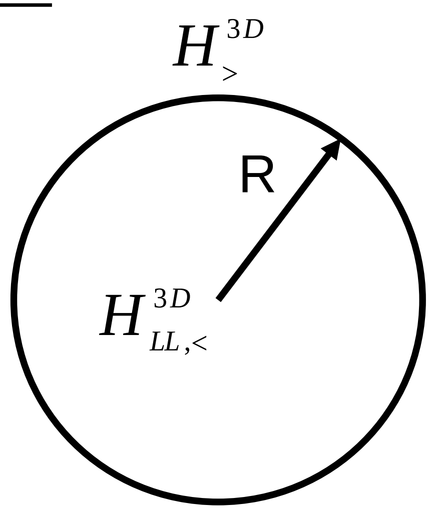
• For our 3D case, the vacuum charge density is plus or minus of the half of the particle density of the non-relativistic LLLs.



• What kind of anomaly?

Helical surface mode of 3D Dirac LL

- The mass of the vacuum outside $M \to +\infty$

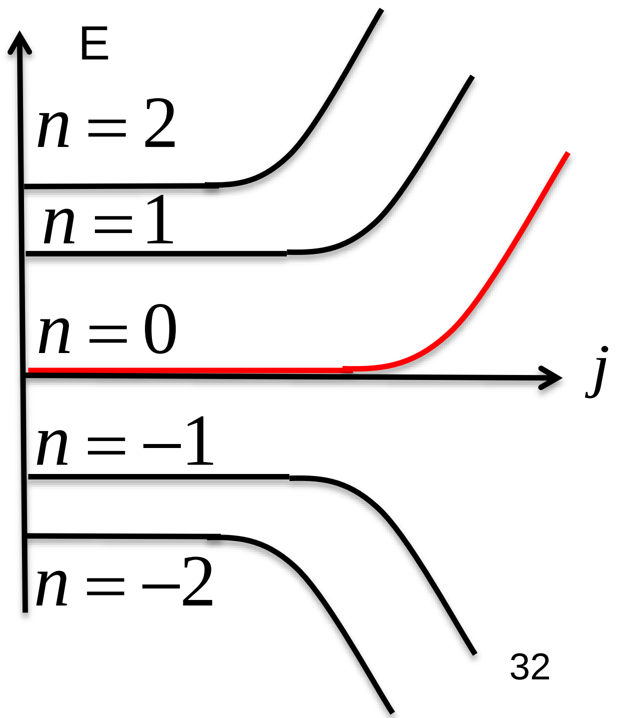


$$H_{<}^{3D} = H_{LL}^{3D} \qquad H_{>}^{3D} = \begin{pmatrix} M & \vec{p}\cdot\vec{\sigma} \\ \vec{p}\cdot\vec{\sigma} & -M \end{pmatrix}$$

- This is the square root problem of the open boundary problem of 3D non-relativistic LLs.

- Each surface mode for n>0 of the non-relativistic case splits a pair surface modes for the Dirac case.

- The surface mode of Dirac zeroth-LL of is singled out. Whether it is upturn or downturn depends on the sign of the vacuum mass.



Conclusions

• We hope the quaternionic analyticity can facilitate the construction of 3D Laughlin state.

• The non-relativistic N-dimensional LL problem is a N-**dimensional harmonic oscillator + spin-orbit coupling.**

• The relativistic version is a square-root problem corresponding to Dirac equation with non-minimal coupling.

• Open questions: **interaction effects; experimental realizations; characterization of topo-properties with harmonic potentials**