Chapter 3 Web Appendix

CALCULATION OF THE NUMBER OF MODES FOR WAVES IN A CAVITY

We wish to solve for the number of standing waves that fall within the frequency range from f to $f + df$ for waves confined to a cubical cavity of side L. Because $N(f)$ is known experimentally to be the same for a cavity of any shape and for walls of any material, we can choose the simplest shape—a cube—and the simplest boundary conditions—waves that vanish at the boundaries. Starting with Maxwell's equations, it can be shown that the electric field obeys a wave equation that may be separated into time-dependent and time-independent parts. The time-dependent equation has a simple sinusoidal solution with frequency ω, and each time-independent component of the electric field satisfies an equation of the type

$$\frac{\partial^2 E_x}{\partial x^2} + \frac{\partial^2 E_x}{\partial y^2} + \frac{\partial^2 E_x}{\partial z^2} + k^2 E_x = 0 \tag{3.40}$$

where

$$E_x = E_x(x, y, z)$$

and

$$k = \frac{2\pi}{\lambda} = \frac{2\pi f}{c}$$

Assuming that $E_x = u(x)v(y)w(z)$, we can separate Equation 3.40 into three ordinary differential equations of the type

$$\frac{d^2 u}{dx^2} + k_x^2 u = 0 \tag{3.41}$$

where

$$k^2 = k_x^2 + k_y^2 + k_z^2$$

Equation 3.41 is the simple harmonic oscillator equation and has the solution

$$u(x) = B \cos k_x x + C \sin k_x x$$

Imposing the boundary condition that E_x or u is 0 at $x = 0$ and at $x = L$ leads to $B = 0$ and $k_x L = n_x \pi$, where $n_x = 1, 2, 3, \ldots$, similar solutions are obtained for $v(y)$ and $w(z)$, giving

$$E_x(x, y, z) = A \sin(k_x x) \sin(k_y y) \sin(k_z z)$$

where

$$k^2 = \frac{\pi^2}{L^2}(n_x^2 + n_y^2 + n_z^2) \tag{3.42}$$

and n_x, n_y, n_z are positive integers.

To determine the density of modes, we interpret Equation 3.42 as giving the square of the distance from the origin to a point in k-space, or "reciprocal" space. It is called reciprocal space because k has dimensions of $(\text{length})^{-1}$. As shown in Figure 3.30, the axes in k-space are k_x, k_y, and k_z. Because $k_x = n_x\pi/L$, $k_y = n_y\pi/L$, and $k_z = n_z\pi/L$, the points in k-space (or modes) are separated by π/L along each axis, and there is one standing wave in k-space per $(\pi/L)^3$ of volume. The number of standing waves, $N(k)$,

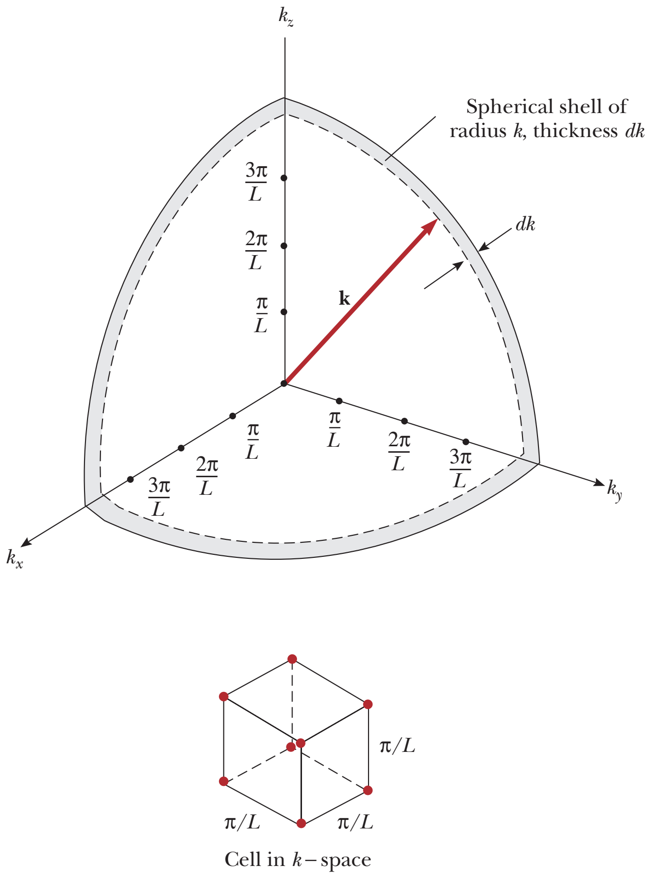

Figure 3.30 A geometrical interpretation of $k^2 = (\pi^2/L^2)(n_x^2 + n_y^2 + n_z^2)$.

having wavenumbers between k and $k + dk$ is then simply the volume of k-space between k and $k + dk$ divided by $(\pi/L)^3$. The volume between k and $k + dk$ is 1/8 the volume of a spherical shell of thickness dk, so that

$$N(k)\,dk = \frac{\frac{1}{2}\pi k^2\,dk}{(\pi/L)^3} = \frac{Vk^2\,dk}{2\pi^2} \tag{3.43}$$

where $V = L^3$ is the cavity volume.

For electromagnetic waves there are two perpendicular polarizations for each mode, so that $N(k)$ in Equation 3.43 must be increased by a factor of 2. Therefore, we have for the number of standing waves per unit volume

$$\frac{N(k)\;dk}{V} = \frac{k^2\;dk}{\pi^2} \tag{3.44}$$

To find $N(f)$ we use $k = 2\pi f/c$ in Equation 3.44 to obtain

$$N(f)\;df = \frac{8\pi f^2}{c^3}\,df \tag{3.45}$$

$N(\lambda)$, the number of modes per unit volume between λ and $\lambda + d\lambda$, may be obtained from Equation 3.45 by using $f = c/\lambda$ to give

$$N(\lambda)\,d\lambda = \frac{8\pi}{\lambda^4}\,d\lambda \tag{3.46}$$